1

Monochromatic and Zero-Sum Sets of Nondecreasing Diameter

Carl R. Yerger[1]

Georgia Institute of Technology

Abstract. Let $k, r, s \in \mathbb{N}$ where $r \geq s \geq 2$. Define $f(s, r, k)$ to be the smallest positive integer n such that for every coloring of the integers in $[1, n]$ there exist subsets S_1 and S_2 such that: (a) S_1 and S_2 are monochromatic (but not necessarily of the same color), (b) $|S_1| = s$, $|S_2| = r$, (c) $\max(S_1) < \min(S_2)$, and (d) $diam(S_1) \leq diam(S_2)$. In this paper, we prove the following:

For $r \geq s \geq 2$, $f(s, r, 2) =$

$$\begin{cases} 5s-3 & \text{if } s = r \\ 4s+r-3 & \text{if } s < r \leq 2s-2 \\ 2s+2r-2 & \text{if } r > 2s-2 \end{cases}$$

For $r \geq s \geq 3$, $f(s, r, \mathbb{Z}) =$

$$\begin{cases} 5s-3 & \text{if } s = r \\ 4s+\max(r, s+\frac{s}{(r,s)}-1)-3 & \text{if } s < r \leq 2s-2 \\ 2s+2r-2 & \text{if } r > 2s-2 \end{cases}$$

For $r \geq s \geq 3$, $f(s, r, 3) =$

$$\begin{cases} 9s-7 & \text{if } r \leq \lfloor\frac{5s-1}{3}\rfloor - 1 \\ 4s+3r-4 & \text{if } \lfloor\frac{5s-1}{3}\rfloor - 1 < r \leq 2s-2 \\ 6s+2r-6 & \text{if } 2s-2 < r \leq 3s-3 \\ 3s+3r-4 & \text{if } r > 3s-3 \ (r \geq s \geq 2) \end{cases}$$

For $r \geq s \geq 3$, $f(s, r, \{\infty\} \cup \mathbb{Z}) =$

$$\begin{cases} 9s-7+\frac{s}{(r,s)}-1 & \text{if } r \leq \lfloor\frac{5s-1}{3}\rfloor - 1 \\ 3s+2r-3+\min(3s-3, s+r-1+\frac{s}{(r,s)}-1) & \text{if } \lfloor\frac{5s-1}{3}\rfloor - 1 < r \leq 3s-3 \\ 3s+3r-4 & \text{if } r > 3s-3 \ (r \geq s \geq 2) \end{cases}$$

We prove that the theorems defining $f(s, r, 2)$ and $f(s, r, 3)$ admit a partial generalization in the sense of the Erdős-Ginzburg-Ziv theorem. This work begins the off-diagonal case of the results of Bialostocki, Erdős, and Lefmann.

Key words. zero-sum, monochromatic, coloring, Erdős-Ginzburg-Ziv

1. Introduction

In 1961, Erdős, Ginzburg and Ziv in [8] proved the following theorem that has been the subject of many recent developments.

* The author was supported by NSF grant DMS0097317. *Email*: cyerger@math.gatech.edu

Theorem 1.1. *Let $m \in \mathbb{N}$. Every sequence of $2m-1$ elements from $\mathbb{Z}$ contains a subsequence of m elements whose sum is zero modulo m.*

Notice that this theorem is a generalization of the pigeonhole principle. Indeed, if the sequence contains only the residues 0 and 1, then this theorem describes the situation of placing $2m-1$ pigeons in 2 holes.

We begin by introducing some notation and definitions. A mapping $\Delta : X \longrightarrow C$, where C is the set of colors, is called a *coloring*. If $C = \{1, \ldots, k\}$, we say that Δ is a k-coloring. If the set of colors is $\mathbb{Z}$, the additive group of the integers, we call Δ a $\mathbb{Z}$-coloring. A set X is called *monochromatic* if $\Delta(x) = \Delta(x')$ for all $x, x' \in X$. In a *$\mathbb{Z}$-coloring* of X, a subset Y of X is called *zero-sum* mod m if $\sum_{y \in Y} \Delta(y) \equiv 0 \bmod m$. If $a, b \in \mathbb{N}$, then $[a, b]$ is the set of integers $\{n \in \mathbb{N} | a \leq n \leq b\}$. For finite $X \subseteq \mathbb{N}$, define the *diameter* of X, denoted by $diam(X)$, to be $diam(X) = \max(X) - \min(X)$. For integers, s, r, let (s, r) be the greatest common factor of s and r. Denote by ∞ an additional color, which does not belong to $\mathbb{Z}$. For brevity and ease of expression, we denote proofs of a parallel nature by parentheses.

Definition 1.2. *Let $f(s, r, k)$ (Let $f(s, r, \mathbb{Z})$) (Let $f(s, r, \{\infty\})$) be the the smallest positive integer n such that for every coloring $\Delta : [1, n] \longrightarrow [1, k]$ ($\Delta : [1, n] \longrightarrow \mathbb{Z}$) ($\Delta : [1, n] \longrightarrow \{\infty\} \cup \mathbb{Z}$), there exist two subsets S_1, S_2 of $[1, n]$, which satisfy:*
(a)S_1 and S_2 are monochromatic (S_1 is zero-sum mod s and S_2 is zero-sum mod r) (S_1 is either ∞-monochromatic or zero-sum mod s and S_2 is either ∞-monochromatic or zero-sum mod r),
(b)$|S_1| = s, |S_2| = r$,
(c) $\max(S_1) < \min(S_2)$, and
(d)$diam(S_1) \leq diam(S_2)$.

In [4] it was shown that $f(m, m, 2) = f(m, m, \mathbb{Z}) = 5m - 3$ and $f(m, m, 3) = f(m, m, \{\infty\} \cup \mathbb{Z}) = 9m - 7$. (Such theorems are known as zero-sum generalizations in the sense of Erdős-Ginzberg-Ziv). Several papers continued the investigations of [4]. The body of work done on this topic is mostly separated into two sections. The first involves coloring the integers with one set, begun in [17] and continued in [1], [5], [15] and [16]. The second involves coloring the integers with two sets, begun in [4] and further considered in [12], [13], [18] and [19]. Initial work on coloring the integers with three sets has begun in [14].

In this paper, we evaluate $f(s, r, 2)$, $f(s, r, \mathbb{Z})$, $f(s, r, 3)$ and $f(s, r, \{\infty\} \cup \mathbb{Z})$ for one of the off-diagonal cases, namely $r \geq s$, as shown in the tables below.

$$\text{For } r \geq s \geq 2,\ f(s, r, 2) =$$

$$\begin{cases} 5s - 3 & \text{if } s = r \\ 4s + r - 3 & \text{if } s < r \leq 2s - 2 \\ 2s + 2r - 2 & \text{if } r > 2s - 2 \end{cases}$$

$$\text{For } r \geq s \geq 3,\ f(s, r, \mathbb{Z}) =$$

$$\begin{cases} 5s - 3 & \text{if } s = r \\ 4s + \max(r, s + \frac{s}{(r,s)} - 1) - 3 & \text{if } s < r \leq 2s - 2 \\ 2s + 2r - 2 & \text{if } r > 2s - 2 \end{cases}$$

$$\text{For } r \geq s \geq 3,\ f(s, r, 3) =$$

$$\begin{cases} 9s-7 & \text{if } r \le \lfloor \frac{5s-1}{3} \rfloor - 1 \\ 4s+3r-4 & \text{if } \lfloor \frac{5s-1}{3} \rfloor - 1 < r \le 2s-2 \\ 6s+2r-6 & \text{if } 2s-2 < r \le 3s-3 \\ 3s+3r-4 & \text{if } r > 3s-3 \ (r \ge s \ge 2) \end{cases}$$

For $r \ge s \ge 3$, $f(s, r, \{\infty\} \cup \mathbb{Z}) =$

$$\begin{cases} 9s-7+\frac{s}{(r,s)}-1 & \text{if } r \le \lfloor \frac{5s-1}{3} \rfloor - 1 \\ 3s+2r-3+\min(3s-3, s+r-1+\frac{s}{(r,s)}-1) & \text{if } \lfloor \frac{5s-1}{3} \rfloor - 1 < r \le 3s-3 \\ 3s+3r-4 & \text{if } r > 3s-3 \ (r \ge s \ge 2) \end{cases}$$

A collection of sets S_i that satisfy conditions (a), (b), (c) and (d) of Definition 1.2 is called a *solution* to $f(s, r, k)$ $(f(s, r, \mathbb{Z}))$, $(f(s, r, \{\infty\} \cup \mathbb{Z}))$. For every positive integer r and s, it turns out that $f(s, r, 2)$ and $f(s, r, 3)$ admit a zero-sum generalization in the sense of Erdős-Ginzburg-Ziv only for some values of s and r. This paper is organized as follows. In Section 2, we present some preliminaries from additive number theory. In Section 3, we prove the values of $f(s, r, 2)$ and $f(s, r, \mathbb{Z})$. In Section 4, the values of $f(s, r, 3)$ and $f(s, r, \{\infty\} \cup \mathbb{Z})$ are determined.

2. Preliminaries from Additive Number Theory

The following generalization of the Erdős-Ginzburg-Ziv (EGZ) theorem was shown in [2], and it will be used in our evaluations of $f(s, r, \mathbb{Z})$, $f(s, r, 3)$ and $f(s, r, \{\infty\} \cup \mathbb{Z})$.

Theorem 2.1. *Suppose that S is a set. Let $m \ge 3$ be an integer, let $\mathbb{Z}_m$ denote the cyclic group on m elements, and let $\Delta : S \to \mathbb{Z}_m$ be a coloring for which $|\Delta(S)| \ge 3$. If $|S| = 2m-2$, then there exist distinct integers $x_1, \ldots, x_m$ such that $\sum_{i-1}^{m} \Delta(x_i) = 0$.*

Next, we state a slightly stronger form of the EGZ theorem [7] used in our determination of $f(s, r, \{\infty\} \cup \mathbb{Z})$.

Theorem 2.2. *Let k and m be positive integers such that $k|m$. If $\Delta : [1, m+k-1] \to \mathbb{Z}_m$, then there exist distinct integers $x_1, \ldots, x_m \in [1, m+k-1]$ such that $\sum_{i=1}^{m} \Delta(x_i) \equiv 0 \bmod k$. Moreover, $m+k-1$ is the smallest number for which the above assertion holds.*

Throughout the paper, we apply Theorem 2.2 to the following situation: Suppose there are a elements all colored by the same residue mod r. This means there are $\frac{s}{(r,s)}$ possible residues mod s for these a elements. Therefore, by Theorem 2.2, it takes at most $s+\frac{s}{(r,s)}-1$ elements to obtain an s-element zero-sum mod s set from the a elements.

Let G be an finite abelian group. If $a_1, \ldots, a_s$ is a sequence S of elements from G, then an *n-set partition* of S is a collection of n nonempty subsequences of S, pairwise disjoint as sequences, such that every term of S belongs to exactly one of the subsequences, and the terms in each subsequence are all distinct, so that the subsequences may be considered sets. Grynkiewicz proved in [9] and [12] the following theorem, which is used in the determination of $f(s, r, \{\infty\} \cup \mathbb{Z})$. The result is an extension of the Cauchy-Davenport theorem.

Theorem 2.3. *Let $a_1, \ldots, a_s$ be a sequence S of elements from an abelian group G of order m with an n-set partition $P = P_1, \ldots, P_n$. Furthermore, let p be the smallest prime*

divisor of m*. Then either:*
(i) there exists an n*-set partition* $A = A_1, A_2, \ldots, A_n$ *of* S *such that:*

$$\left|\sum_{i=1}^{n} A_i\right| \geq \min\{m,\ (n+1)p,\ |S| - n + 1\};$$

furthermore, if $n' \geq \frac{m}{p} - 1$ *is an integer such that* P *has at least* $n - n'$ *cardinality one sets and if* $|S| \geq n + \frac{m}{p} + p - 3$*, then we may assume there are at least* $n - n'$ *cardinality one sets in* A*, or* (ii)*:*
(a) *there exists* $\alpha \in G$ *and a nontrivial proper subgroup* H_a *of index* a *such that all but at most* $a - 2$ *terms of* S *are from the coset* $\alpha + H_a$*; and*
(b) *there exists an* n*-set partition* $A_1, A_2, \ldots, A_n$ *of the subsequence of* S *consisting of terms from* $\alpha + H_a$ *such that* $\sum_{i=1}^{n} A_i = n\alpha + H_a$*.*

3. The 2-Coloring and its EGZ Generalization

In this section, we will evaluate $f(s,r,2)$ and $f(s,r,\mathbb{Z})$. First, we introduce notion helpful in the proof. For $c \in C$, where $\Delta^{-1}(c) \neq \emptyset$, we denote $first(c) = \min\{x \in X | \Delta(x) = c\}$ and $last(c) = \max\{x \in X | \Delta(x) = c\}$. A coloring $\Delta : [1,n] \longrightarrow C$ will be identified by the strings $\Delta(1)\Delta(2)\cdots\Delta(n)$, and we use x^i to denote the string, $xx\cdots x$, of length i. First, we evaluate $f(s,r,2)$.

Lemma 3.1. *Let* r *and* s *be positive integers with* $r \geq s \geq 2$ *and* $r \leq 2s - 2$*. If* $\Delta : [1, 2s + r - 2] \longrightarrow \mathbb{Z}$ *is a coloring, then either,*
(i) there exists an r*-element zero-sum mod* r *subset* $R \subseteq [1, 2s+r-2]$ *such that* $diam(R) \geq 2s - 2$ *or*
(ii) there exists a zero-sum solution to $f(s,r,\mathbb{Z})$*.*

Proof. Let $I_1 = [1,r], I_2 = [r+1, 2s-2]$, and $I_3 = [2s-1, 2s+r-2]$. Since $|I_1 \cup (I_3 \setminus \{2s-1\})| = |(I_1 \setminus \{r\}) \cup I_3| = 2r - 1$, by the pigeonhole principle, it follows that both $I_1 \cup (I_3 \setminus \{2s-1\})$ and $(I_1 \setminus \{r\}) \cup I_3$ contain r-element monochromatic sets, say R_1 and R_2, respectively. If $R_1 \cup I_1 \neq \emptyset$ and $R_1 \cap (I_3 \setminus \{2s-1\}) \neq \emptyset$, then $diam(R_1) \geq 2s - 2$ whence (i) follows. Similarly, if $R_2 \cap (I_1 \setminus \{r\}) \neq \emptyset$ and $R_2 \cap I_3 \neq \emptyset$, then $diam(R_2) \geq 2s-2$, and (i) follows. Therefore, we can assume that I_1 and I_3 are r-element monochromatic sets. Therefore, condition (ii) follows by taking $[1,s]$ and I_3. □

Theorem 3.2. *Let* r *and* s *be positive integers where* $r \geq s \geq 2$*. If* $r \leq 2s - 2$*, then* $f(s,r,2) = 4s + r - 3$*.*

Proof. The coloring $\Delta : [1, 4s + r - 4] \longrightarrow \{0,1\}$ given by the string:

$$01^{s-1}0^{s-1}1^{2s-2}0^{r-1}$$

implies that $f(s,r,2) > 4s + r - 4$. Next we show that $f(s,r,2) \leq 4s + r - 3$. Let $\Delta : [1, 4s+r-3] \longrightarrow \{0,1\}$ be an arbitrary coloring. By the pigeonhole principle, the interval $[1, 2s-1]$ contains an s-element monochromatic subset K such that $diam(K) \leq 2s - 2$. Shifting the interval $[2s, 4s+r-3]$ to the interval $[1, 2s+r-2]$ and applying Lemma 3.1 completes the proof. □

We now will compute $f(s, r, \mathbb{Z})$. First, consider a special case of the 2-coloring that allows us to give a simple example of how $f(s, r, 2) \neq f(s, r, \mathbb{Z})$.

Theorem 3.3. *Let r and s be positive integers where $r \geq s \geq 3$ and r, s are coprime. If $r \geq 2s - 2$, then $f(s, r, \mathbb{Z}) = 6s - 4$.*

Proof. The coloring $\Delta : [1, 6s - 5] \longrightarrow \{1, 2\}$ given by the string:

$$\begin{aligned} 01^{s-1}0^{s-1}1^{s-1}0^{s-1}1^{s-1}0^{s-1} &\mod s \\ 01^{s-1}0^{s-1}1^{2s-2}0^{2s-2} &\mod r \end{aligned}$$

implies that $f(s, r, 2) > 2s + 2r - 3$. By Theorem 2.1, it follows that the interval $[1, 2s - 1]$ contains an s-element zero-sum mod s subset S_1 with $diam(S_1) \leq 2s - 2$, and that the interval $[2s, 6s-4]$ contains an r-element zero-sum mod r subset S_2 with $diam(S_2) > 2s-2$. Hence, S_1 and S_2 complete the proof. □

As we can see, since $f(s, r, 2) = 4s + r - 3 \neq 6s - 5$ (for $r < 2s - 2$), in this case, we do not have a generalization in the sense of Erdős-Ginzburg-Ziv. However, when $r > 2s - 2$, an EGZ generalization exists.

Theorem 3.4. *Let r and s be positive integers where $r \geq s \geq 2$. If $r > 2s - 2$, then $f(s, r, 2) = f(s, r, \mathbb{Z}) = 2s + 2r - 2$.*

Proof. The coloring $\Delta : [1, 2s + 2r - 3] \longrightarrow \{1, 2\}$ given by the string

$$\begin{aligned} 01^{s-1}0^{s-1}1^{r-1}0^{r-1} &\mod s \\ 01^{s-1}0^{s-1}1^{r-1}0^{r-1} &\mod r \end{aligned}$$

implies that $f(s, r, 2) > 2s + 2r - 3$. By Theorem 1.1, it follows that the interval $[1, 2s - 1]$ contains an s-element zero-sum mod s subset S_1 with $diam(S_1) \leq 2s - 2$, and that the interval $[2s, 2s + 2r - 2]$ contains an r-element zero-sum mod r subset S_2 with $diam(S_2) > 2s-2$. Hence, S_1 and S_2 complete the proof, since $f(s, r, 2) \geq f(s, r, \mathbb{Z})$ holds trivially. □

We expand on the previous definitions of $first(k)$ and $last(k)$ in the following manner. Let $x_1 < x_2 < \ldots < x_n$ be the integers colored by c in Y. Then, for integers i and j such that $1 \leq i \leq j \leq n$, we use the notation $first_i^j(c, Y)$ to denote $\{x_i, x_{i+1}, \ldots, x_j\}$. Likewise $first_i(c, Y) = \{x_i\}$, and $first(c, Y) = first_1(c, Y)$. Similarly, we define $last_i^j(c, Y) = \{x_{n-i+1}, x_{n-i}, \ldots, x_{n-j+1}\}$, $last_i(c, Y) = \{x_{n-i+1}\}$, and $last(c, Y) = last_1(c, Y)$. If we do not specify Y, then it is assumed that Y is the entire interval. We now will compute $f(s, r, \mathbb{Z})$ for the remaining cases.

Lemma 3.5. *Let r and s be positive integers with $r \geq s \geq 3, r \leq 2s - 2$, and $(r, s) > 1$ (and $s + \frac{s}{(r,s)} - 1 > r$). If $\Delta : [1, 2s + r - 2]$ ($[1, 3s + \frac{s}{(r,s)} - 3]$) $\longrightarrow \mathbb{Z}$ is a coloring, then either:*
(i) there exists an r-element zero-sum mod r subset $R \subseteq [1, 2s + r - 2]$ ($[1, 3s + \frac{s}{(r,s)} - 3]$) such that $diam(R) \geq 2s - 2$ or
(ii) there exists a zero-sum solution to $f(s, r, \mathbb{Z})$.

Proof. Let $I_1 = [1, r], I_2 = [r + 1, 2s - 2]$, and $I_3 = [2s - 1, 2s + r - 2]$. ($I_3 = 2s - 1 \cup last_1^{r-1}([1, 3s + \frac{s}{(r,s)} - 3])$). Since $|I_1 \cup (I_3 \setminus \{2s - 1\})| = |(I_1 \setminus \{r\}) \cup I_3| = 2r - 1$, in view of Theorem 1.1, it follows that both $I_1 \cup (I_3 \setminus \{2s - 1\})$ and $(I_1 \setminus \{r\}) \cup I_3$

contain an r-element zero-sum mod r set, say R_1 and R_2, respectively. If $R_1 \cap I_1 \neq \emptyset$ and $R_1 \cap (I_3 \setminus \{2s-1\}) \neq \emptyset$, then $diam(R_1) \geq 2s-2$, whence (i) follows. Similarly, if $R_2 \cap (I_1 \setminus \{r\}) \neq \emptyset$, then $diam(R_2) \geq 2s-2$, and (i) follows. Therefore, we can assume that I_1 and I_3 are r-element zero-sum mod r sets.

If $\Delta(I_1) \cap (\Delta(I_3 \setminus \{2s-1\})) \neq \emptyset$, then by replacing an element $x \in I_1$ with some element $y \in I_3 \setminus \{2s-1\}$, where $\Delta(x) = \Delta(y)$, then we obtain a set R with $diam(R) \geq 2s-2$, whence (i) follows. Hence, $\Delta(I_1) \cap \Delta(I_3) = \emptyset$. Furthermore, if I_1 is monochromatic mod r, then it takes at most $s + \lfloor \frac{s}{(r,s)} \rfloor - 1$ elements to obtain an s-element zero-sum mod s set S_2. So if $r \geq s + \lfloor \frac{s}{(r,s)} \rfloor - 1$, S_1 and I_3 satisfy condition (ii).

(Otherwise, let $J_2 = \{2s-1\}$ and let $J_3 = last_1^{r-1}(I_3)$. Notice that since $r < s + \lfloor \frac{s}{(r,s)} \rfloor - 1$ that S_2 does not exist yet in this case (hence the reason for the extended argument). First suppose that there are only two residues colored by $\mathbb{Z}$ mod r. There cannot be only one residue mod r colored by $\mathbb{Z}$ since $\Delta(I_1) \cap \Delta(I_3) = \emptyset$. Let A be the residue mod r used to color I_1, and let B be the other residue mod r. Notice that $last(A) \leq 2s-2$ else condition (i) holds. Thus, the interval $[2s-1, 3s + \frac{s}{(r,s)} - 3]$ is colored entirely by residue B mod r, whence residue B mod r contains at least r elements. Thus, the interval $[1, s + \frac{s}{(r,s)} - 1]$ is colored entirely by residue B. Thus, there exists an s-element zero-sum mod s set S_3 with $diam(S_3) \leq s + \frac{s}{(r,s)} - 1$. Condition (ii) then holds by taking S_3 and $J_2 \cup J_3$. Thus, there exists at least three residues of $\mathbb{Z}$ mod r.

Now we will apply Theorem 2.3 in order to finish this section of the proof. Let A and B be the residues mod r as defined in the previous paragraph. Let $a = first(\mathbb{Z})$ and $b = first_2(\mathbb{Z})$. Let $y = last_2(\mathbb{Z})$ and $z = last(\mathbb{Z})$. We must satisfy the conditions of Theorem 2.3, by 1) creating an $(r-2)$-set partition of $[1, 3s + \frac{s}{(r,s)} - 3]$ with $r-3$ sets of cardinality 2 and one set of cardinality 3, and 2) fixing two elements that are not part of the set partition, in this case a and z, unless a and z contain the only residues not in $A \cup B$. In this case, fix b instead of a or y instead of z. In any case, the diameter of this set described by Theorem 2.3 is at least $2s-2$. Hence, we can apply Theorem 2.3 to this case.

Therefore, there exists an $(r-2)$-set partition of $[1, 3s + \frac{s}{(r,s)} - 3]$ with $m-3$ sets of cardinality 2 and one set of cardinality 3 as well as two fixed elements. (Notice we have at least $r+1$ elements not colored by A and thus enough for our set partition because if the only elements where a third color occurs was in I_3, then we can apply Theorem 2.1, and this case will be discussed after the argument based on Theorem 2.3.) We apply Theorem 2.3 to observe that 1) either there is an r-element zero-sum mod r set R_4 that contains the two fixed elements, and $r-2$ other elements of the set partition, one from each set, or 2) all but at most $r-2$ of the elements colored by $\mathbb{Z}$ are colored by elements from the same coset $(a\mathbb{Z}_m + \alpha)$ of $\mathbb{Z}_m$. If the first conclusion occurs, $diam(R_4) \geq 2s-2$, whence condition (i) follows. In the latter case, Theorem 2.3 implies that any subset of cardinality $(r + \frac{r}{a} - 1 + a - 2) \leq \lceil \frac{3r}{2} \rceil - 1$ must contain an r element zero-sum mod r set. Hence, consider the set $R_5 = first_1^{\lceil \frac{3r}{4} \rceil}(\mathbb{Z}, [1, 3s + \frac{s}{(r,s)} - 3]) \cup last_1^{\lceil \frac{3r}{4} \rceil}(\mathbb{Z}, [1, 3s + \frac{s}{(r,s)} - 3])$. Within R_5, there exists an r-element zero-sum mod r set R_6 with $diam(R_6) \geq 3s-3$. Condition (i) holds.)

Thus, $\Delta(I_1)$ is not monochromatic mod r. Let $M = [1, r-1]$, and let $M' = [2, r]$. Let $N = [2s, 2s+r-2] (N = J_3)$. Since I_1 is not monochromatic mod r, then either M or M', say M, is colored by at least two residues. Let $P = M \cup N$. Since $|\Delta(M)| \geq 2$ and since $\Delta(M) \cap \Delta(N) = \emptyset$, it follows that $|\Delta(P)| \geq 3$. Hence, since $|M| = r-1$ and $|N| = r-1$,

then by Theorem 2.1, it follows that there exists an r-element zero-sum mod r set $P' \subset P$ with $P' \cap M \neq \emptyset$ and $P' \cap N \neq \emptyset$, whence condition (i) follows.

□

Theorem 3.6. *Let r and s be positive integers where $r \geq s \geq 3, r \leq 2s-2$ and $(r,s) > 1$. If $r \geq s + \frac{s}{(r,s)} - 1$, then $f(s,r,2) = f(s,r,\mathbb{Z}) = 4s + r - 3$.*

Proof. The coloring $\Delta : [1, 4s + r - 4] \longrightarrow \{1, 2\}$ given by the string:

$$01^{s-1}0^{s-1}1^{2s-2}0^{r-1} \text{ mod } s$$
$$01^{s-1}0^{s-1}1^{2s-2}0^{r-1} \text{ mod } r$$

implies that $f(s,r,\mathbb{Z}) > 4s + r - 4$. Next we show that $f(s,r,\mathbb{Z}) \leq 4s + r - 3$. Let $\Delta : [1, 4s + r - 3] \longrightarrow \mathbb{Z}$ be an arbitrary coloring. By Theorem 1.1, it follows that the interval $[1, 2s-1]$ contains an s-element zero-sum mod s subset K with $diam(K) \leq 2s-2$. Shifting the interval $[2s, 4s+r-3]$ to the interval $[1, 2s+r-2]$ and applying Lemma 3.5 completes the proof, since $f(s,r,2) \geq f(s,r,\mathbb{Z})$ holds trivially. □

Theorem 3.7. *Let r and s be positive integers where $r \geq s \geq 3, r \leq 2s-2$ and $(r,s) > 1$. If $r < s + \frac{s}{(r,s)} - 1$, then $f(s,r,\mathbb{Z}) = 5s + \frac{s}{(r,s)} - 4$.*

Proof. The coloring $\Delta : [1, 5s + \frac{s}{(r,s)} - 5] \longrightarrow \mathbb{Z}$ given by the string:

$$10^{s-1}1^{s-1}0^{s-1}2^{\frac{s}{(r,s)}-1}0^{s-\frac{s}{(r,s)}}1^{s+\frac{s}{(r,s)}-2} \text{ mod } s$$
$$10^{s-1}1^{s-1}0^{2s-2}1^{s+\frac{s}{(r,s)}-2} \text{ mod } r$$

implies that $f(s,r,\mathbb{Z}) > 5s + \frac{s}{(r,s)} - 5$. Next we show that $f(s,r,\mathbb{Z}) \leq 5s + \frac{s}{(r,s)} - 4$. Let $\Delta : [1, 5s + \frac{s}{(r,s)} - 4] \longrightarrow \mathbb{Z}$ be an arbitrary coloring. By Theorem 1.1, the interval $[1, 2s-1]$ contains an s-element zero-sum mod s subset K such that $diam(K) \leq 2s-2$. Shifting the interval $[2s, 5s + \frac{s}{(r,s)} - 4]$ to the interval $[1, 3s + \frac{s}{(r,s)} - 3]$ and applying Lemma 3.5 completes the proof. □

4. The 3-Coloring and its EGZ Generalization

Let $\delta = r - (2s-2)$ for $r \geq 2s-2$. Otherwise, $\delta = 0$. First, we will determine $f(s,r,3)$.

Lemma 4.1. *Suppose r and s are positive integers with $r \geq s \geq 3, r \leq \lfloor \frac{5s-1}{3} \rfloor - 1$ ($\lfloor \frac{5s-1}{3} \rfloor - 1 \leq r \leq 3s-3$) and let A be a subset of $[1, 6s-5]$ with $|A| \geq 5s-4$. (Let A be a subset of $[1, s+3r-2-\delta]$ with $|A| \geq 3r-1-\delta$). If $\Delta :\longrightarrow \{1,2\}$ is a coloring then either:*
(i) there exists a monochromatic r-element subset $S \subseteq A$ with $diam(S) \geq 3s-3$ or
(ii) there exists a solution to $f(s,r,3)$.

Proof. Let $I_1 = first_1^r(A, [1, 6s-5])$ ($first_1^r(A, [1, s+3r-2-\delta])$) and $I_3 = last_1^{\max(2s-1, r-1)}(A, [1, 6s-5])$ ($I_3 = last_1^{\max(2s-1,r-1)}(A, [1, s+3r-2-\delta])$). Since $|I_1 \cup I_3| \geq 2r+1$, it follows that there exists a monochromatic set S in $I_1 \cup I_3$ with cardinality $r+1$. If $S \cap I_1 \neq \emptyset$ and $S \cap I_3 \neq \emptyset$, then $diam(S) \geq 3s-3$, and assertion (i) follows, since we can construct a monochromatic set S' from S with $|S'| = r$ and $diam(S') \geq 3s-3$. Otherwise, since $|I_1| < r+1$, it follows that I_3 contains S. If $\Delta(I_1) \cap \Delta(I_3) \neq \emptyset$, we can replace an element

$x \in S$ with a corresponding element $y \in I_1$, where $\Delta(x) = \Delta(y)$, to construct a set T with $diam(T) \geq 3s-3$. Assertion (i) is satisfied, since we can construct a set T' from T with $|T'| = r$ and $diam(T') \geq 3s-3$. Thus assume that $\Delta(I_1) \cap \Delta(I_3) = \emptyset$. Hence, I_1 and I_3 are each monochromatic. Call an element $x \in [\min(I_1), \max(I_1)]$ a *hole* of I_1 if $x \notin I_1$. There are at most $s-1$ holes. Hence, $diam(I_1) \leq 2s-2$. Let S'' be any r-element subset of I_3 with $diam(S'') \geq diam(I_3)$. The sets I_1 and S'' satisfy assertion (ii), thus completing the proof. □

Lemma 4.2. *Let r, s be positive integers where $r \geq s \geq 3$, with $r \geq \lfloor \frac{5s-1}{3} \rfloor - 1$ ($\lfloor \frac{5s-1}{3} \rfloor - 1 < r \leq 3s-3$). If $\Delta : [1, 6s-5]([1, s+3r-2-\delta]) \longrightarrow \{1,2,3\}$ is a coloring, then either:*
(1) there exists a monochromatic r-element subset $S \subseteq [1, 6s-5]$ ($S \subseteq [1, s+3r-2-\delta]$) with $diam(S) \geq 3s-3$ or
(ii) there exists a solution to $f(s, r, 3)$.

Proof. Let $\Delta : [1, 6s-5]$ $([1, s+3r-2-\delta]) \longrightarrow \{1,2,3\}$ be given. If one color occurs at most $s-1$ times, and if A is the set of integers colored by the other two colors, then $|A| \geq 5s-4$ ($|A| \geq 3r-1-\delta$). Hence, the proof is satisfied by Lemma 4.1. Thus, each color occurs at least s times.

Suppose that at most one color, call it the third color, denoted by 3, contains at least r elements. Then, there are at least $6s-5-2(\lfloor \frac{5s-1}{3} \rfloor - 2) \geq 2\lceil \frac{4s+1}{3} \rceil - 1 \geq s+r$ elements colored by the third color. (There are at least $s+3r-2-\delta-2(r-1) \geq s+r-\delta$ elements colored by the third color.) There can be at most $3s-3$ integers in the interval $[first(3), last(3)]$, else condition (i) holds.

Suppose that $r < \lfloor \frac{3s}{2} \rfloor - 1$. Since the first two colors each contain fewer than r elements, the third color consists of at least $6s-5-2(\lfloor \frac{3s}{2} \rfloor - 1) - 1 \geq 3s-2$ elements. Consider the set $B = first_1^{r-1}(3) \cup last(3)$. Since $diam(B) \geq 3s-2$, condition (i) holds, and hence we can assume that $r \geq \lfloor \frac{3s}{2} \rfloor - 1$. There are at least $s+r-1 \geq \lfloor \frac{5s}{2} \rfloor$ elements colored by the third color. Let $C = first_1^{s}(3)$. Since there are at most $3s-3$ elements colored by the third color else condition (i) holds, $diam(C) \leq \lceil \frac{3s}{2} \rceil - 2$. Since $r \geq \lfloor \frac{3s}{2} \rfloor - 1$ condition (ii) holds by taking the sets C and $D = first_{s+1}^{s+r-1}(3) \cup last(3)$. Thus, at least two of the three colors must contain at least r elements.

Let w.l.o.g. $\Delta(1) = 1$. Let the next new color be denoted 2 and the last new color be denoted 3. Suppose that the first s elements are monochromatic. In this case, there can be at most $r-1$ more elements of each color else condition (ii) holds. Since $s+3(\lfloor \frac{5s-1}{3} \rfloor - 2) < 6s-5$ $(s+3(r-1)-1-\delta < s+3r-2-\delta)$, the next element in the set after $r-1$ elements of each color in any order will obtain condition (ii). Hence, there exists some element of $[1, s]$ not colored by color 1. To complete the proof of the lemma, we will consider four different cases based upon the cardinality of colors 1 and 2.

Case 1: Assume there are at most $r-1$ elements of color 1 and color 2.

Since there must be at least two colors with at least r elements, we have a contradiction.

Case 2: Assume there are at least r elements of color 1 and at most $r-1$ elements of color 2.

There must be at least r elements of color 3 because there must be two or more colors with at least r elements. We will now show that the interval $[1, 2s-1]$ is only colored by colors 1 and 2. Suppose that $first(3) \leq \lceil \frac{4s-2}{3} \rceil - 1$. Since color 3 is colored by at least r elements, $last(3) \leq \lceil \frac{13s-2}{3} \rceil - 5$ else condition (i) holds via the set $E = first_1^{r-1}(3) \cup last(3)$. Hence, the interval $[\lceil \frac{13s-2}{3} \rceil - 4, 6s-5]$ $([\lceil \frac{13s-2}{3} \rceil - 4, s+3r-2-\delta])$, consists of at least $\lceil \frac{5s}{3} \rceil - 1 \geq r$ $(3r - \lfloor \frac{10s}{3} \rfloor + 3 - \delta \geq r)$ elements colored by 2, which is a contradiction.

We will repeat this argument two more times by shifting the location of some elements colored by 2 from the beginning of the interval to the end of the interval. Suppose that $\lceil\frac{4s-2}{3}\rceil - 1 < first(3) \le \lceil\frac{5s-2}{3}\rceil - 1$. Hence, the interval $[\lceil\frac{14s-2}{3}\rceil - 4, 6s-5]$ ($[\lceil\frac{14s-2}{3}\rceil - 4, s+3r-2-\delta]$) is colored entirely by 2, which contains at least $\lceil\frac{4s-2}{3}\rceil - 1$ ($3r-3s+2-\delta$) elements. There are at least $\lceil\frac{s-2}{3}\rceil$ elements colored by 2 in the interval $[1, \lceil\frac{4s-2}{3}\rceil - 1]$, else we obtain a monochromatic set $F = first_1^s(1, [1, \lceil\frac{4s-2}{3}\rceil - 1])$ where $diam(F) \le \lceil\frac{4s-2}{3}\rceil - 2 \le r$, assuming $r > \lceil\frac{4s-2}{3}\rceil - 2$, thereby forcing condition (i). Notice that if $r \le \lceil\frac{4s-2}{3}\rceil - 2$, then there are already too many elements colored by 2 in $[\lceil\frac{14s-2}{3}\rceil - 4, 6s-5]$ ($[\lceil\frac{14s-2}{3}\rceil - 4, s+3r-2-\delta]$). Hence, condition (ii) holds by taking E and F.

Next, suppose that $\lceil\frac{5s-2}{3}\rceil - 1 < first(3) \le 2s-1$. In this case, the interval $[1, \lceil\frac{5s-2}{3}\rceil - 1]$ requires $\lceil\frac{2s-2}{3}\rceil$ elements colored by 2 else condition (ii) holds, assuming that $r \ge \lceil\frac{5s-2}{3}\rceil - 1$. Otherwise, if $r \le \lceil\frac{5s-2}{3}\rceil - 1$, we know there are at least $r-s+1$ elements colored by 2 in $[1, \lceil\frac{5s-2}{3}\rceil - 1]$, else we can force condition (ii) with $F' = first_1^s(1) \cup last_1^r(3)$. Since the interval $[\lceil\frac{15s-2}{3}\rceil - 4, 6s-5]$ ($[\lceil\frac{15s-2}{3}\rceil - 4, s+3r-2-\delta]$) is colored entirely by 2, we require an additional $s-1$ ($3r - \lfloor\frac{8s}{3}\rfloor + 3 - \delta$) elements for a total of $r-s+1+s-1 = r$ ($3r - \lfloor\frac{10s-2}{3}\rfloor + 2 - \delta \ge r$) elements colored by 2, which is a contradiction. Hence, the interval $[1, 2s-1]$ is colored entirely by 1 and 2.

By Theorem 1.1, there exists an s-element monochromatic set G in $[1, 2s-1]$ with $diam(G) \le 2s-2$. (Thus, if $r \ge 2s-1$, condition (ii) follows). This implies that there exists an interval in $[2s, 6s-5]$ ($[2s, s+3r-2-\delta]$) of diameter at most $2s-3$ consisting of all the elements colored by 3. Since the interval $[3s-2, 6s-5]$ ($[3s-2, s+3r-2-\delta]$) is colored only by 2 and 3, there must be at least s elements colored by 2 in this interval. Since there are at least r elements colored by 3, $diam(G) \ge r$, else condition (ii) holds. Hence, there are at least $r-s+1$ elements colored by 2 in the interval $[1, 2s-1]$. This implies that there are at least $s+r-s+1 = r+1$ elements colored by 2, a contradiction, which completes this case.

Case 3: Assume there are at least r elements of color 1 and color 2.

Since $first(1) = 1, last(1) \le 3s-3$, else condition (i) follows. Similarly, since $first(2) \le s$, then $last(2) \le 4s-4$. Thus, the interval $[4s-3, 6s-5]$ ($[4s-3, s+3r-2-\delta]$) is colored entirely by color 3. It follows that there is a monochromatic s-element set $H \subset [1, 2s-1]$ with $diam(H) \le 2s-2$, which along with $H' = first_1^{r-1}(3) \cup last(3)$, yields condition (ii).

Case 4: Assume there are at most $r-1$ elements of color 1 and at least r elements of color 2.

Since $first(2) \le s$, then $last(2) \le 4s-4$, else condition (i) follows. Hence the interval, $[4s-3, 6s-5]$ ($[4s-3, s+3r-2-\delta]$) is entirely colored by 1 and 3. Since at least two of the colors must contain at least r elements, there must be at least r elements colored by color 3. Using the same argument as in Case 2 except that we condition on the location of the elements colored by 1, the interval $[1, 2s-1]$ is colored entirely by colors 1 and 2. Hence, there exists a monochromatic s-element set $I = first_1^s(1, [1, 2s-1])$ or $first_1^s(2, [1, 2s-1])$, such that $diam(I) \le 2s-2$. If there are $2s-1$ or more elements colored by 3, then condition (ii) follows by taking I and $J = first_1^{r-1}(3) \cup last(3)$. Thus, there are at most $2s-2$ elements colored by 3. In addition, it follows that there exists an interval of diameter at most $2s-3$ consisting of all the elements colored by 3. (Notice that condition (ii) follows using I and J if $r \ge 2s-1$.)

Suppose that $2s-1 < first(3) \le 3s-2$. Since there are at least r elements colored by 3, there must be at least $r-s+1$ elements colored by 1 in $[1, 2s-1]$ else condition (ii)

holds by taking $K = first_1^s(1,[1,2s-1])$ or $first_1^s(2,[1,2s-1])$ and $L = first_1^{r-1}(3) \cup last(3)$. Since $first(3) \le 3s-2, last(3) < 5s-4$. Hence, the interval $[5s-4, 6s-5]$ ($[5s-4, s+3r-2-\delta]$), consisting of at least $s-1$ elements is colored entirely by color 1. A total of $(s-1)+(r-s+1) = r$ elements are colored by 1, a contradiction. Hence, the interval $[1, 3s-2]$ is colored entirely by 1 and 2.

Next, suppose $r \le \lfloor \frac{3s}{2} \rfloor - 2$. If there are at least $\lfloor \frac{5s}{2} \rfloor - 1$ elements colored by 2, then we can construct $M = first_1^s(2)$ with $diam(M) \le \lceil \frac{3s}{2} \rceil - 2$. Condition (ii) holds with M and $N = first_{s+1}^{s+r-1}(2) \cup last(2)$, where $diam(N) \ge \lfloor \frac{3s}{2} \rfloor - 2$. Thus, there are at most $\lfloor \frac{5s}{2} \rfloor - 2$ elements colored by 2, and there are at least $\lceil \frac{3s}{2} \rceil - 1$ elements colored by 1. (Recall $r < 2s-1$ and $|\Delta^{-1}(3)| \le 2s-2$.) But this means that there are at least r elements colored by 1, a contradiction. Therefore $r > \lfloor \frac{3s}{2} \rfloor - 2$.

Suppose that $3s-2 < first(3) \le \lfloor \frac{10s}{3} \rfloor - 3$. Let $T = first_1^{r-1}(3) \cup last(3)$, where $diam(T) = t$, and let $\alpha = 2s-2-t$. Then the interval, $[\lfloor \frac{16s}{3} \rfloor - 5 - \alpha, 6s-5]$ ($[\lfloor \frac{16s}{3} \rfloor - 5 - \alpha, s+3r-2-\delta]$), is colored entirely by 1. In this interval, there are at least $\lceil \frac{2s}{3} \rceil + \alpha + 1$ ($\lceil \frac{2s}{3} \rceil + \alpha + 1 + 3(r - \lfloor \frac{5s-1}{3} \rfloor) - 1 - \delta$) elements colored by 1. Thus, there are at most $s-2-\alpha$ elements left that can be colored by 1. Consider the interval $[2, 2s-\alpha]$. Since $\Delta(1) = 1$, there are $s-3-\alpha$ elements of color 1 remaining before a contradiction occurs. We must also avoid creating an s-element monochromatic set of diameter t or less else condition (ii) holds. Thus, there must be at least $t-s+1 = s-1-\alpha$ elements colored by 1 in the interval $[2, 2s-\alpha]$. However, this is more than $s-3-\alpha$ elements, and so we have obtained a contradiction. Thus, $first(3) > \lfloor \frac{10s}{3} \rfloor - 5$.

Next, suppose that $\lfloor \frac{10s}{3} \rfloor - 3 < first(3) \le \lfloor \frac{11s}{3} \rfloor - 3$. Then the interval $[\lfloor \frac{17s}{3} \rfloor - 5 - \alpha, 6s-5]$ ($[\lfloor \frac{17s}{3} \rfloor - 5 - \alpha, s+3r-2-\delta]$) is colored entirely by 1 and the interval $[1, \lfloor \frac{10s}{3} \rfloor - 3]$ is colored entirely by 1 and 2. In this interval, there are at least $\lfloor \frac{s}{3} \rfloor + \alpha + 1$ ($\lceil \frac{s}{3} \rceil + \alpha + 1 + 3(r - \lfloor \frac{5s-1}{3} \rfloor) - 1 - \delta$) elements colored by 1. Thus, there are at most $\lfloor \frac{4s}{3} \rfloor - 2 - \alpha$ elements left that can be colored by 1 before we have a contradiction. The diameter of the elements colored by color 2 is at most $3s-3$, so there are $\lfloor \frac{s}{3} \rfloor$ elements that are forced to be colored by 1 outside the interval $[first(2), last(2)]$ in $[1, \lfloor \frac{10s}{3} \rfloor - 3]$. Subtracting those elements colored by 1 leaves $s-2-\alpha$ integers that are allowed to be colored by 1. Inside $[first(2), last(2)]$, we must avoid creating an s-element monochromatic set of diameter t or less, else condition (ii) holds. Thus, there must be at least $t-s+2 = s+\alpha$ elements colored by 1 in the interval $[first(2), first(2)+2s-2-\alpha]$. But there aren't enough elements colored by 1 to satisfy these constraints. Thus, $first(3) > \lfloor \frac{11s}{3} \rfloor - 3$.

Next suppose that $\lfloor \frac{11s}{3} \rfloor - 3 < first(3) \le 4s-3$. Hence the interval $[6s-5-\alpha, 6s-5]$ ($[6s-5-\alpha, s+3r-2-\delta]$) is colored entirely by 1. In this interval, there are at least $\alpha+1$ ($\alpha + 1 + 3(r - \lfloor \frac{5s-1}{3} \rfloor) - 1 - \alpha$) elements colored by 1. thus, there are at most $\lfloor \frac{5s}{3} \rfloor - 2 - \alpha$ elements that are colored by 1. The diameter of the elements colored by color 2 is at most $3s-3$, so there are $\lfloor \frac{2s}{3} \rfloor$ elements that must be colored by 1 outside this interval. Subtracting those elements of 1 leaves $s-2-\alpha$ integers colored by 1 before we obtain a contradiction. We must avoid creating an s-element monochromatic set of diameter t or less in the interval $[first(2), first(2)+2s-2-\alpha]$, else condition (ii) holds. Thus, there must be at least $t-s+2 = s-\alpha$ elements colored by 1 in this interval. But we only have $s-2-\alpha$ elements colored by 1 left before we obtain a contradiction. Thus, $first(3) > 4s-3$.

Finally, suppose that $first(3) > 4s-3$. Then the intervals $[4s-3, first(3)-1]$ and $[last(3)+1, 6s-5]$ ($[last(3)+1, s+3r-2-\delta]$) are colored entirely by 1. In the interval $[4s-3, 6s-5]$ ($[4s-3, s+3r-2-\delta]$) least, there are at least α ($\alpha + 3(r - \lfloor \frac{5s-1}{3} \rfloor) - 1 - \delta$)

elements colored by 1. Thus, there are at most $\lfloor \frac{5s}{3} \rfloor - 2 - \alpha$ elements left that may be colored by 1. The diameter of the elements colored by color 2 is at most $3s-3$, so there are (at least) $s-1$ elements that must be colored by 1 in $[1, 4s-3]$ outside $[first(2), last(2)]$ because the interval $[1, 4s-3]$ is colored entirely by 1 and 2. Subtracting those elements of 1 leaves at most $\lfloor \frac{2s}{3} \rfloor - 1 - \alpha$ integers colored by 1. We must avoid creating an s-element monochromatic set of length t, else condition (ii) holds. Thus, there must be at least $t - s - 2 = s - \alpha$ elements colored by 1 in the interval, $[first(2), first(2)+2s-2-\alpha]$. However, there aren't enough elements colored by 1 to satisfy these constraints without obtaining a contradiction to our assumptions. Thus, our proof of the lemma is complete. □

Theorem 4.3. *Let r and s be positive integers where $r \geq s \geq 3$. If $r \leq \lfloor \frac{5s-1}{3} \rfloor - 1$ then $f(s, r, 3) = 9s - 7$.*

Proof. The coloring $\Delta : [1, 9s-8] \longrightarrow \{1, 2, 3\}$ given by the string:

$$12^{s-1}1^{s-2}3^{s-1}12^{s-1}1^{s-1}2^{s-1}1^{s-1}3^{2s-2}$$

implies that $f(s, r, 3) > 9s-8$. Next we show that $f(s, r, 3) \leq 9s-7$. Let $\Delta : [1, 9s-7] \longrightarrow \{1, 2, 3\}$ be an arbitrary coloring. By Theorem 1.1, the interval $[1, 3s-2]$ contains an s-element monochromatic subset S such that $diam(S) \leq 3s-3$. Shifting the interval $[3s-1, 9s-7]$ to the interval $[1, 6s-5]$ and applying Lemma 4.2 completes the proof for $r \geq s \geq 3$. □

Theorem 4.4. *Let r and s be positive integers where $r \geq s \geq 3$. If $\lfloor \frac{5s-1}{3} \rfloor - 1 < r \leq 2s-2$ then $f(s, r, 3) = 4s + 3r - 4$.*

Proof. The coloring $\Delta : [1, 4s+3r-5] \longrightarrow \{1, 2, 3\}$ given by the string:

$$12^{s-1}1^{s-2}3^{s-1}12^{s+r-1}1^{r-1}3^{r-1}$$

implies that $f(s, r, 3) > 4s + 3r - 5$. Next we show that $f(s, r, 3) \leq 4s + 3r - 4$. Let $\Delta : [1, 4s+3r-4] \longrightarrow \{1, 2, 3\}$ be an arbitrary coloring. By Theorem 1.1, the interval $[1, 3s-2]$ contains an s-element monochromatic subset S such that $diam(S) \leq 3s-3$. Shifting the interval $[3s-1, 4s+3r-4]$ to the interval $[1, s+3r-2]$ and applying Lemma 4.2 completes the proof for $r \geq s \geq 3$. □

Theorem 4.5. *Let r and s be positive integers where $r \geq s \geq 3$. If $2s-2 < r \leq 3s-3$ then $f(s, r, 3) = 6s + 2r - 6$.*

Proof. The coloring $\Delta : [1, 6s+2r-7] \longrightarrow \{1, 2, 3\}$ given by the string:

$$12^{s-1}1^{s-2}3^{s-1}12^{3s-3}1^{r-1}3^{r-1}$$

implies that $f(s, r, 3) > 6s + 2r - 7$. Next we show that $f(s, r, 3) \leq 6s + 2r - 6$. Let $\Delta : [1, 6s+2r-6] \longrightarrow \{1, 2, 3\}$ be an arbitrary coloring. By Theorem 1.1, the interval $[1, 3s-2]$ contains an s-element monochromatic subset S such that $diam(S) \leq 3s-3$. Shifting the interval $[3s-1, 6s+2r-6]$ to the interval $[1, 4s+2r-4]$ and applying Lemma 4.2 completes the proof for $r \geq s \geq 3$. □

Theorem 4.6. *Let r and s be positive integers where $r \geq s \geq 2$. If $r > 3s-3$ then $f(s, r, 3) = 3s + 3r - 4$.*

Proof. The coloring $\Delta : [1, 3s+3r-5] \longrightarrow \{1,2,3\}$ given by the string:

$$12^{s-1}1^{s-2}3^{s-1}12^{r-1}1^{r-1}3^{r-1}$$

implies that $f(s,r,3) > 3s+3r-5$. By the pigeonhole principle, it follows that the interval $[1, 3s-2]$ contains an s-element monochromatic subset S_1 with $diam(S_1) \leq 3s-3$, and that the interval $[3s-1, 3s+3r-4]$ contains an r-element monochromatic subset S_2 with $diam(S_2) \geq 3s-3$. Hence, S_1 and S_2 complete the proof. □

Now we will determine $f(s,r,\{\infty\} \cup \mathbb{Z})$.

Lemma 4.7. *Let r and s be positive integers where $r \geq s \geq 3$. If $\Delta : [1, 6s-5+\frac{s}{(r,s)}-1] \longrightarrow \{\infty\} \cup \mathbb{Z}$ ($\Delta : [1, s+2r-4+\min(3s-3, s+r-1+\frac{s}{(r,s)}-1)] \longrightarrow \{\infty\} \cup \mathbb{Z}$) is a coloring, then either:*
(i) there exists an r-element subset $S \subseteq [1, 6s-5+\frac{s}{(r,s)}-1]$ ($S \subseteq [1, 2r-1+\min(3s-3, s+r-1+\frac{s}{(r,s)}-1)]$) such that
(a) S is either ∞-monochromatic or zero-sum mod r and
(b) $diam(S) \geq 3s-3$ or
(ii) there exists a solution to $f(s,r,\{\infty\} \cup \mathbb{Z})$.

Proof. We partition the proof into three cases, each considering a different cardinality of the sets $\Delta^{-1}(\mathbb{Z})$ and $\Delta^{-1}(\infty)$.
Case 1: Suppose $|\Delta^{-1}(\mathbb{Z})| \leq 3s-3$.

This implies that $|\Delta^{-1}(\infty)| \geq 3s-2$. There exists an r-element subset $S \subseteq \Delta^{-1}(\infty)$ such that S is ∞-monochromatic and $diam(S) \geq 3s-3$. Condition (i) of the lemma follows.
Case 2: Suppose that $|\Delta^{-1}(\infty)| \leq r-1$.
Case 2.1: First, consider the case when $r \leq \lfloor \frac{3s}{2} \rfloor - 1$ or $2s-2 \leq r \leq 3s-3$.

In this case we do not have to apply Theorem 2.3. Let $I_1 = first_1^r(\mathbb{Z})$ and let $I_3 = last_1^r(\mathbb{Z})$. Also let $a = last(\mathbb{Z}, I_1), b = first(\mathbb{Z}, I_3), S_1 = (I_1 \setminus \{a\}) \cup I_3$ and $S_2 = I_1 \cup (I_3 \setminus \{b\})$. Since $|S_1| = |S_2| = 2r-1$, by Theorem 1.1, there exist r-element zero-sum mod r sets X, Y in S_1, S_2, respectively. If $X \cap (I_1 \setminus \{a\}) \neq \emptyset$ and $X \cap I_3 \neq \emptyset$, then $diam(X) \geq 3s-3$, and assertion (i) of the lemma is satisfied. Else, $X = I_3$ since $|I_1 \setminus \{a\}| \leq r$. Repeating the above argument for Y, either (i) follows or $Y = I_1$. Hence, I_1 and I_3 are r-element zero-sum mod r sets.

Consider the coloring of the sets I_1 and I_3. If $\Delta(I_1) \cap \Delta(I_3) \neq \emptyset$, then replace an element $u \in I_1$ with an element $v \in I_3$, where $\Delta(u) = \Delta(v)$ mod r to construct an r-element zero-sum mod r set S with $diam(S) \geq 3s-3$. Hence, assume that $\Delta(I_1) \cap \Delta(I_3) = \emptyset$. Suppose that at least one of the sets I_1 and I_3 is not monochromatic. Without loss of generality, suppose that I_1 is colored by at least two residues mod r. Let $M = first_1^{r-1}(\mathbb{Z}, I_1)$, and let $M' = last_1^{r-1}(\mathbb{Z}, I_1)$. Either M or M', say M, is colored by at least two residues. Let $N = first_1^{r-1}(\mathbb{Z}, I_3)$. Consider the set $A = M \cup N$. It is colored by at least three residues mod r since M is colored by at least two residues and N is colored by at least one residue. By Theorem 2.1, there exists an r-element zero-sum mod r set $A' \subset A$ with $diam(A') \geq 3s-3$, whence condition (i) follows. Thus, both I_1 and I_3 are monochromatic mod r.

Let $J_1 = I_1 \setminus \{a\}, J_2 = \{first(\mathbb{Z}, [first(\mathbb{Z}) + 3s-2, last(\mathbb{Z})])\}$, and $J_3 = last_1^{r-1}(I_3)$. Within $J_1 \cup J_2 \cup J_3$, there are $2r-1$ elements, and so there exists an r-element zero-sum mod r set B. If B is contained in $B_1 = J_1 \cup J_2$, $B_2 = J_1 \cup J_3$ or $B_3 = J_1 \cup J_2 \cup J_3$,

with the additional condition that each of the named component sets in each union must be nonempty, (i.e. for B_1, both J_1 and J_2 must be nonempty) then condition (i) follows. If not, then B is contained in $J_2 \cup J_3$. If $r \geq s + \frac{s}{(r,s)} - 1$, then in I_1, there exists an s-element zero-sum mod s set C. If α counts the number of elements colored by ∞ in $[first(B), last(B)]$, then $diam(B) \leq s + \frac{s}{(r,s)} - 2 + \alpha$. If $r \geq s + \frac{s}{(r,s)} - 1$, then condition (ii) follows by taking B and C, where $diam(C) \geq \lceil \frac{3s}{2} \rceil - 3 + \frac{s}{(r,s)} + \alpha$.

(If $r < s + \frac{s}{(r,s)} - 1$, then first suppose that there are only two residues colored by $\mathbb{Z}$. There cannot be only one residue colored by $\mathbb{Z}$ since $\Delta(I_1) \cap \Delta(I_3) = \emptyset$. Let 1 be the residue mod r used to color I_1 and let 2 be the residue mod r used to color I_3. This implies that both residue 1 and residue 2 contain at least r elements mod r. Notice that $last(1) < J_2$ and $first(2) > first_{3s-r-1}(\mathbb{Z})$ else condition (i) holds. This implies that the interval $[1, first_{3s-r-1}(\mathbb{Z})]$, is colored entirely by ∞ or 1 mod r. Also, the interval $[J_2, last(\mathbb{Z})]$ is colored entirely by residue 2 mod r. If α is the number of elements colored by ∞ in $[first(\mathbb{Z}), first_{3s-r-1}(\mathbb{Z})]$, then there exists an s-element zero sum mod s set D where $diam(D) \leq s + \frac{s}{(r,s)} - 1 + \alpha \leq 3s - r - 1 + \alpha$. Since there are at least $3s - r - 1 + \alpha$ elements colored by residue 2 mod r, condition (ii) holds by taking D and $first(2) \cup last_1^{r-1}(2)$.)

Case 2.2: Next, consider the case when $\lfloor \frac{3s-1}{2} \rfloor - 1 \leq r \leq 2s - 2$.

In this case, we must apply Theorem 2.3. Let $a = first(\mathbb{Z}), b = first_2(\mathbb{Z}), y = last_2(\mathbb{Z})$ and $z = last(\mathbb{Z})$. Since $|\Delta^{-1}(\infty)| \leq r - 1$, then $|\Delta^{-1}(\mathbb{Z})| \geq 6s - r - 5 + \frac{s}{(r,s)} - 1$ $(|\Delta^{-1}(\mathbb{Z})| \geq r + \omega$, where $\omega = \min(3s - 3, s + r - 1 + \frac{s}{(r,s)} - 1))$. Consider the interval $S = [a, z]$. Let c be the residue of $\mathbb{Z}$ that colors the most number of integers mod r, and let d be the residue that colors the second most number of integers mod r.

Suppose there are only two residues that are colored by $\mathbb{Z}$ mod r. If residue d had fewer than r elements mod r, then residue c would consist of at least $s + \lfloor \frac{5s}{3} \rfloor + \frac{s}{(r,s)} - 2$ $(1 + \omega)$ elements and condition (ii) would follow. Thus, both residues must contain at least r elements mod r. This means that the diameters of the set of elements containing each residue is at most $3s - 3$. Let 1 be the residue that appears first and let 2 be the residue that appears second, making $first(1) < first(2)$. The set $A = first_1^{s + \frac{s}{(r,s)} - 1}(1)$ contains an s-element zero-sum mod s set, A'. Recall that there are at least $s + 2r - 2 + \frac{s}{(r,s)}$ elements colored by 1 or 2. Notice that $first(2)$ must occur after $first_{s + \frac{s}{(r,s)} - 1}(1)$, else condition (i) follows because there would be more than 3s - 3 elements occurring after both residues mod r have already appeared. Thus $first(2) > first_{s + \frac{s}{(r,s)} - 1}(1)$. But then condition (ii) follows by taking the sets A' and $B = first(2) \cup last_1^{r-1}(2)$. To see that $diam(A) \leq diam(B)$, if α is the number of elements colored by ∞ within the interval that contains A', there must be at least $s + 2r - 1 + \frac{s}{(r,s)} - 1 - (3s - 3 - \alpha) = 2r - 2s + 2 + \frac{s}{(r,s)} + \alpha \geq s + \frac{s}{(r,s)} - 1$. Hence, there must exist some integer that is colored by a third residue mod r.

Case 2.2.1 Suppose there are at least $r + 1$ elements of $\mathbb{Z}$ not colored by c mod r.

We must satisfy the conditions of Theorem 2.3 so we can produce an r-element zero-sum mod r set E with $diam(E) \geq 3s - 3$ if conclusion (i) of Theorem 2.3 holds. If there are at least two elements not colored by c or d, we satisfy the conditions of Theorem 2.3, by 1) creating an $(r-2)$-set partition of $\Delta(S)$ with $r - 3$ sets of cardinality 2 and one set of cardinality 3, and 2) fixing two elements that are not part of the set partition, in this case a and z. This holds unless the only two elements not colored by c or d are located at a and z, whence we cannot force a set with diameter $z - a$. In this case, fix b and z, and the diameter of this set described by Theorem 2.3 is still at least $3s - 3$. Hence, we

can apply Theorem 2.3 to this case if there are at least two elements of $\mathbb{Z}$ not colored by c or d mod r.

Suppose that there is only one element of $\mathbb{Z}$ colored by something other than c or d mod r. In this case, fix a and z for the set partition, except if either a or z is the third color. If a is the third color, fix b. Likewise, if z is the third color, y. In any of these three instances, the diameter of the r-element zero-sum mod r set that may be satisfied by Theorem 2.3 is still greater than $3s-3$. Hence we can apply Theorem 2.3 to this case, and will do this after Case 2.2.2.

Case 2.2.2: Suppose that all but r elements of $\mathbb{Z}$ are colored by c mod r. Notice that if all but $r-1$ elements of $\mathbb{Z}$ are colored by c, color c would consist of at least $s+\lfloor\frac{5s}{3}\rfloor+\frac{s}{(r,s)}-2$ $(1+\omega)$ elements, and condition (ii) would follow. Therefore, there exists a $c\in\mathbb{Z}$ such that $|\Delta^{-1}(c)\cup S| = 6s-2r-5+\frac{s}{(r,s)} \geq s+\frac{s}{(r,s)}+\lfloor\frac{5s}{3}\rfloor-3$ $(|\Delta^{-1}(c)\cup S|=\omega)$ mod r. Thus, if there are at least $\lceil\frac{s}{3}\rceil-\frac{s}{(r,s)}+1$ elements not colored by c in $[first(c), last(c)]$, condition (i) follows since we have constructed an r-element zero-sum mod r set E with $diam(E)\geq 3s-3$. Let $F = first_1^{s+\frac{s}{(r,s)}-1}(c)$. There exists an s-element zero-sum mod s set F' such that $diam(F')\leq\lceil\frac{4s}{3}\rceil\leq r$. Take the other $r-1$ elements colored by c and combine them with $r-1$ elements colored by $\mathbb{Z}$ having at least two different colors than c mod r. These differently colored residues exist since we have already considered the case when there are only two residues of $\mathbb{Z}$ mod r. Hence, in this set, there exists an r-element zero-sum mod r set G. If all the elements of G are after F', then condition (ii) holds. If there exist elements of G with F' or before F', we can construct an $(r-2)$-set partition in the following manner: Fix a and z. If either of these elements is the only element that is colored by the third color, then replace it with b or y, respectively. After fixing the above elements, the remaining elements form a $(2r-3)$-element $(r-2)$-set partition. We are left with the case when $r<4$. This only occurs when both r and s are 3 and this is considered in [4], completing Case 2.2.2.

Therefore, in both Case 2.2.1 and Case 2.2.2, there exists an $(r-2)$-set partition of $\Delta(S)$ with $m-3$ sets of cardinality 2 and one set of cardinality 3 as well as two fixed elements. We apply Theorem 2.3 to observe that 1) either there is an r-element zero-sum mod r set H that contains the two fixed elements and $r-2$ other elements of the set partition, one from each set, or 2) all but at most $r-2$ of the elements colored by $\mathbb{Z}$ are colored by elements from the same coset $(a\mathbb{Z}_m+\alpha)$ of $\mathbb{Z}_m$. If the first conclusion occurs, $diam(H)\geq 3s-3$, whence condition (i) follows. In the latter case, Theorem 2.2 implies that any subset of cardinality $(r+\frac{r}{a}-1+a-2)\leq\lceil\frac{3s}{2}\rceil-1$ must contain an r-element zero-sum mod r set. Hence, consider the set $J = first_1^{\lceil\frac{3r}{4}\rceil}(\mathbb{Z})\cup last_1^{\lceil\frac{3r}{4}\rceil}(\mathbb{Z})$. Within J, there exists an r-element zero-sum mod r set K with $diam(K)\geq 3s-3$. Condition (i) holds, and the proof for this case is complete.

Case 3: $|\Delta^{-1}(\mathbb{Z})|\geq 2s-1$ and $|\Delta^{-1}(\infty)|\geq r$.

Case 3.1: Suppose there are only one or two residues colored by $\mathbb{Z}$ mod r.

Clearly, if there is only one residue colored by $\mathbb{Z}$ mod r, we are finished. Now, suppose that there are precisely two residues of $\mathbb{Z}$ mod r. Without loss of generality, let color 1 be the first residue colored by $\mathbb{Z}$ mod r and let 2 be the second color mod r.

Case 3.1.1: Suppose that $first(\infty) < first(1)$. Suppose that $first(1) > s$. Then condition (ii) holds by taking $first_1^s(\infty)$ along with an r-element zero-sum mod r set from $last_1^{2r-1}(\mathbb{Z})$. Therefore, let $first(1)\leq s$. Let $A_1 = first_1^{r-1}(\mathbb{Z}), A_2 = last_1^{r-1}(\mathbb{Z})$ and $A_3 = first(\mathbb{Z},[first(\mathbb{Z})+3s-3, last(\mathbb{Z})])$. Notice that if A_3 is contained in A_2, then

$first(\mathbb{Z}) > s$, which is a contradiction. By Theorem 1.1, there exists an r-element zero-sum mod r set A. If A is contained using only and precisely C_1 and C_2, C_1 and C_3 or elements from all three sets, then condition (i) holds.

Otherwise, A contains elements from only C_2 and C_3. Suppose that C_1 or C_3 is not monochromatic mod r. If this were so, we could interchange an element of C_1 with an element in C_2 or C_3, and condition (i) would follow. Therefore, C_1 and C_3 are monochromatic mod r. Let $\alpha = first(1)$. Then $first(2) \leq 3s-3+\alpha$. So $|\Delta^{-1}| \geq 6s-5+\frac{s}{(r,s)}-1-(3s-4+\alpha) \geq 3s-1-\alpha+\frac{s}{(r,s)}-1$. However, notice that in the interval $[1, 2s+\frac{s}{(r,s)}-2]$, there exists either an s-element ∞-monochromatic set or an s-element zero-sum mod s set of diameter at most $2s-2+\frac{s}{(r,s)}-1$. Call this set B. The maximum value of α is s, so $|\Delta^{-1}(2)| \geq 2s-1+\frac{s}{(r,s)}-1$, and so the minimum diameter of C, an r-element zero-sum mod r set, colored entirely by 2 is $2s-2+\frac{s}{(r,s)}-1$. Therefore, B and C satisfy condition (ii), completing this subcase.

Case 3.1.2: Suppose that $first(1) < first(\infty)$. Now, suppose that $first(\infty) > 2s-1$. Then there are at most $2s-1$ colored by ∞, and $last(\infty) - first(\infty) \leq 2s-2$. By the pigeonhole principle, one of the two residues of $\mathbb{Z}$ mod r must contain at least r elements. Call this residue R_1 and call the other residue R_2, which may or may not contain r elements.

Step 1: If there is only one residue mod r in $[1, s+\frac{s}{(r,s)}-1]$, then condition (i) or (ii) holds using only residues of $\mathbb{Z}$ when $r < s+\frac{s}{(r,s)}-1$. Otherwise, condition (ii) holds by taking D, the s-element zero-sum mod s set in $[1, \frac{s}{(r,s)}-1]$ and E, the r-element ∞-monochromatic set in $first_1^r(\infty)$. Therefore, residues R_1 and R_2 mod r occur before $s+\frac{s}{(r,s)}-1$. We can then conclude that $last(R_1) \leq 4s-4+\frac{s}{(r,s)}$. Therefore, the interval $[4s-3+\frac{s}{(r,s)}, 6s-6+\frac{s}{(r,s)}]$ $([4s-3+\frac{s}{(r,s)}, \max(last(\mathbb{Z}), last(\infty))])$ is colored entirely by ∞ and R_2.

Step 2: Suppose that $2s \leq first(\infty) \leq 3s-3$. There are at most $2s-2$ elements colored by ∞, else condition (ii) would hold. Also, $last(\infty) \leq 5s-6$. Let $\gamma = 2s-3-(last(\infty) - first(\infty))$. Notice that there are at least $s+\frac{s}{(r,s)}+\gamma$ elements colored by R_2 after $last(\infty)$. Therefore there can be at most $r-s-\frac{s}{(r,s)}$ elements colored by R_2 in the rest of the interval else condition (i) holds. However, we know that there are at least $2s-1-\gamma-(s+\frac{s}{(r,s)}-1) = s-\frac{s}{(r,s)}-\gamma$ elements colored by R_2 in $[1, 2s-1]$, else condition (ii) holds using an s-element zero-sum mod s set contained in $[1, 2s-1]$ using elements only from R_1 and an r-element ∞-monochromatic set. But this means there are at least $s+\gamma+\frac{s}{(r,s)}+s-\gamma-\frac{s}{(r,s)} = 2$ elements colored by R_2 and condition (ii) holds. Notice that if $r > 2s$, then there are more than $2s-2$ elements colored by ∞ and condition (ii) holds using an s-element zero-sum mod s set in the interval $[1, 2s-1]$ and the r element ∞-monochromatic set that is of diameter at least $2s$.

Step 3: Suppose that $3s-2 \leq first(\infty)$. Again, there are at most $2s-2$ elements colored by ∞, else condition (ii) would hold. Let $\gamma = 2s-3-(last(\infty) - first(\infty))$. Outside of the intervals $[first(R_1), last(R_1)]$ and $[first(\infty), last(\infty)]$, there are at least $s+\frac{s}{(r,s)}-1+\gamma$ elements colored by R_2. Inside the interval $[first(R_1), last(R_1)]$, there are an additional $s-\frac{s}{(r,s)}-\gamma+1$, else condition (ii) holds by applying an analogous argument from the previous step. This means there are at least $2s$ elements colored by R_2 and condition (i) holds if $r < 2s$ and the last element is not colored by ∞. If $r > 2s$, then there are more than $2s-2$ elements colored by ∞, a contradiction. If the last element is colored by ∞,

condition (i) still holds since $first(R_2) \leq s + \frac{s}{(r,s)} - 1$ and $last(R_2) \geq 4s - 3 + \frac{s}{(r,s)} - 1$, meaning the diameter of this r-element zero-sum mod r set has diameter at least $3s - 3$.

Step 4: Suppose that $s + \frac{s}{(r,s)} - 1 < first(\infty) \leq 2s - 1$. Again, both R_1 and R_2 must be contained in the interval $[1, s + \frac{s}{(r,s)} - 1]$. Therefore, the interval $[5s-3, \max(last(\mathbb{Z}), last(\infty)]$, is colored entirely by R_2. Thus, if there are at least r elements of R_2, condition (i) holds. Therefore, there are fewer than r elements colored by R_2.

Consider two subcases. First, suppose that $last(\infty) - first(\infty) \geq first(\infty) - 1$. Let $\lambda = first(\infty) - (s + \frac{s}{(r,s)} - 1)$. This is the least number of elements colored by R_2 before $first(\infty)$. In addition, there are an additional $5s-3-(3s-3)-first(\infty) = 2s - first(\infty)$ elements colored by R_2 which are after $[first(\infty), last(\infty)]$ but before $5s - 3$. Therefore there are at least $s - 1 + \frac{s}{(r,s)} - 1 + first(\infty) - (s + \frac{s}{(r,s)} - 1) + 2s - first(\infty) = 2s$ elements colored by R_2, and condition (i) follows.

Now, suppose that $last(\infty) - first(\infty) < first(\infty) - 1$. This means that $last(\infty) - first(\infty) \leq 2s - 2$. the interval $[4s - 3 + \frac{s}{(r,s)} - 1, \max(last(\mathbb{Z}), last(\infty))]$ is colored entirely by R_2, and condition (i) holds using the residues of R_2 mod r.

Step 5: Suppose that $2 \leq first(\infty) \leq \frac{s}{(r,s)} - 1$. Consider three cases. In all cases, suppose that there is at least one color with r elements mod r, and call this residue R_3. Call the other residue, which may or may not contain r elements mod r, R_4. If both R_3 and R_4 are present before $first(\infty)$, then the last $2s - 2$ elements are colored by R_4. Then condition (i) holds using the elements in R_4. (If $r \geq 2s - 1$, then the last $4s - 3$ elements are colored by R_4 and condition (i) holds.)

Now suppose that R_3 is the only residue colored before $first(\infty)$. Again, the last $2s - 2$ elements must be colored by R_4. Therefore, the interval $[1, 3s - 3 + \frac{s}{(r,s)} - 1]$ is colored entirely by R_3 and ∞. Suppose that $first(\infty) \leq s - 1$. Then $last(\infty) \leq 4s - 4$. Thus, the interval $[4s - 3, \max(last(\mathbb{Z}), last(\infty))]$ is colored entirely by R_4. However, in the interval $[1, 2s - 1 + \frac{s}{(r,s)} - 1]$, there either exists an s-element zero-sum mod s set or an s-element ∞-monochromatic set. This set has diameter at most $2s - 2 + \frac{s}{(r,s)} - 1$. Similarly, in $[4s - 3, \max(last(\mathbb{Z}), last(\infty))]$, there exists an r-element zero-sum mod r set with diameter at least $2s - 2 + \frac{s}{(r,s)} - 1$. Therefore, condition (ii) holds.

So let $s \leq first(\infty) \leq s + \frac{s}{(r,s)} - 1$. Notice that if $first(R_4) \leq 4s - 4$, then condition (ii) will hold in the following manner: There exists an r-element zero-sum mod r set with diameter at least $2s - 2 + \frac{s}{(r,s)} - 1$ in the interval $[4s - 3, \max(last(\mathbb{Z}), last(\infty))]$. Also, the interval $[3s - 2, 4s - 4]$ must be colored entirely by ∞, else condition (i) would hold using elements of R_3. In order to avoid a condition (ii) contradiction, the interval $[2s - 2 - \frac{s}{(r,s)} + 1, 3s - 3]$ must be colored entirely by R_3. But then the intervals $[1, s - 1]$ and $[2s - 2 - \frac{s}{(r,s)} + 1, 2s - 1 + \frac{s}{(r,s)} - 1]$ contains an s-element zero-sum mod $s$set with diameter $2s - 2 + \frac{s}{(r,s)} - 1$. This forces condition (ii) and we are finished with this case.

Finally, let R_4 be the only residue colored before $first(\infty)$. Suppose that $2s - 1 + \frac{s}{(r,s)} - 1 \leq first(R_3) \leq 3s$. This means that there exists either an s-element zero-sum mod s set or an s-element ∞-monochromatic set of diameter at most $2s - 1 + \frac{s}{(r,s)} - 1$. Therefore, $last(R_3) \leq 5s - 1 + \frac{s}{(r,s)} - 1$. In addition, let $\beta = 3s - first(R_3)$. So there are at least $s - 4 + \beta$ elements colored by R_4. Also, there are at least $s - 1 + \frac{s}{(r,s)} - 1$ elements colored by R_4 in the interval $[1, 2s - 1 + \frac{s}{(r,s)} - 1]$, else condition (ii) holds. If $\beta \geq 4$, then condition (i) holds using r elements of residue R_4. Thus, $3s - 3 \leq first(R_3) \leq 3s$. Suppose that we select either an s-element ∞-monochromatic or zero-sum mod r set with

minimum diameter $s-1+\gamma$ from $[1, 2s-1+\frac{s}{(r,s)}-1]$. Therefore, there are at most $3s-3-\gamma$ elements colored by ∞, else condition (i) holds. Furthermore, there are at most $s-1+\gamma$ elements colored by R_3 else condition (ii) holds. This means there are at least $2s-2+\frac{s}{(r,s)}-1$ elements colored by R_4, and condition (i) holds.

Now suppose that R_4 does not have r elements. Let $3s \leq first(R_3)$. Then from the previous argument, we see that R_4 must have r elements, so now suppose that R_3, R_4 and ∞ each contain r elements. Therefore $last(R_4) \leq 3s-3$. Now, recall that the interval $[1, 2s-1+\frac{s}{(r,s)}-1]$ contains only R_4 and ∞. Therefore, there exists an s-element ∞-monochromatic or zero-sum mod s set with diameter at most $2s-2+\frac{s}{(r,s)}-1$. Thus, $last(R_3) - first(R_3) \leq 2s-3+\frac{s}{(r,s)}-1$. This means that the interval $[3s-2, 4s-3]$ is colored entirely by ∞. Hence the interval $[2s-2-\frac{s}{(r,s)}+1, 3s-3]$ must be colored entirely by R_4. But this forces the interval $[s-\frac{s}{(r,s)}+1, 2s-1-\frac{s}{(r,s)}+1]$ to be colored entirely by ∞. However, condition (i) holds by taking $first(\infty) \cup last_1^{r-1}(\infty)$. This completes the case.

Let $a = first(\mathbb{Z})$ and $b = first_2(\mathbb{Z})$. Let $y = last_2(\mathbb{Z})$ and $z = last(\mathbb{Z})$. Consider the interval $S = [a, z]$. We must satisfy the conditions of Theorem 2.3, by 1) creating an $(r-2)$-set partition with $r-3$ sets of cardinality 2 and one set of cardinality 3, and 2) fixing two elements that are not part of the set partition. Let c be the residue of $\mathbb{Z}$ that colors the most integers.

Case 3.2 Suppose there are fewer than r elements colored by c.

There are at least $2\lfloor\frac{5s-1}{3}\rfloor - 3$ $(2r-1+\omega-\omega')$ elements colored by $\mathbb{Z}$, since there can be at most $\omega' = \max(s+r-2-\delta, s+\lfloor\frac{5s-1}{3}\rfloor-1)$ elements colored by ∞, else condition (ii) holds, by taking $first_1^s(\infty)$ and $first_{s+1}^{s+r-1}(\infty) \cup last(\infty)$ as the s and r-element ∞ monochromatic sets, respectively.

First, suppose there are at least $2r$ elements colored by $\mathbb{Z}$. If this is the case, then there are at least $r+1$ elements colored by $\mathbb{Z}$ not colored by c. Fix a and z to be outside of the $(r-2)$-set partition, except if in the interval $[b, y]$ there are two colors (not including ∞) each containing $r-1$ elements. In this case, we can force b and z to be elements of the potential r-element zero-sum mod r set guaranteed in a condition of Theorem 2.3 by fixing them. The diameter of this set is at least $3s-3$.

Next, consider the case when there are exactly $2r-1$ elements colored by $\mathbb{Z}$. then there are exactly $s+r-1$ elements colored by ∞. There must be exactly $r-1$ elements colored by a single residue, in this case c, else there are at least $r+1$ elements not colored by c and we can satisfy the conditions of Theorem 2.3 in the following manner. Fix a and z to be outside of the $(r-2)$-set partition except if in the interval $[b, z]$ there are only two colors in the elements of $\mathbb{Z}$. If this occurs, interchange b with a or z with y in the exact manner as in the previous paragraph. Hence, there are exactly $r-1$ elements colored by c. Let d be the residue with the second largest number of elements. If there are at least two elements of $\mathbb{Z}$ colored by neither c or d, we will construct an $(r-2)$-set partition of diameter at least $z-b \geq 3s-3$ or $y-a \geq 3s-3$. Fix either a and z, b and z or b and y, and let the remaining elements form a set partition that satisfies Theorem 2.3. If the desired set partition cannot be accomplished, there is only one element of $\mathbb{Z}$ colored by neither c or d and we must not fix this element or fix two elements of color c or fix two elements of color d, else the desired set partition does not exist. The minimum diameter of a set that avoids these conditions, fixes two elements, and produces an $(r-2)$-set partition that we desire is $\lfloor\frac{3r}{2}\rfloor - 2$. By Theorem 2.3, we either obtain a $\lfloor\frac{3r}{2}\rfloor - 2$ diameter zero-sum mod

r set, or else other conditions that will be discussed later hold. We finish this case by describing what happens when we do obtain the $\lfloor \frac{3r}{2} \rfloor - 2$ diameter zero-sum mod r set derived from Theorem 2.3.

Suppose there exists a $\lfloor \frac{3r}{2} \rfloor - 2$ diameter zero-sum mod r set R. (If $r \geq 2s-1$, condition (i) follows.) In order for there to be exactly $2r-1$ elements colored by $\mathbb{Z}$, $r \geq \lfloor \frac{5s-1}{3} \rfloor - 1$ since otherwise $6s - 5 + \frac{s}{(r,s)} - 1 \geq s + 3r - 2$, and condition (ii) follows since there are at least $s + r$ elements colored by ∞. Hence, we have constructed an r-element zero-sum mod r set R where $diam(R) \geq \lceil \frac{15s-1}{6} \rceil - 4$. Hence, there are at most $\lfloor \frac{3s+1}{6} \rfloor$ elements colored by ∞, within $[first(R), last(R)]$, else condition (i) follows. Similarly, there are at most $2s-r-2$ elements colored by $\mathbb{Z}$ in the interval $[first(\infty), last(\infty)]$ else (i) holds by taking $first_1^s(\infty)$ and $first_{s+1}^{s+r-1}(\infty) \cup last(\infty)$ as the s and r-element ∞-monochromatic sets, respectively.

If $first(\infty) < first(R)$, either (i) or (ii) follows, depending on the placement of the elements colored in R. If there exists an element of R before $first_s(\infty)$, then condition (i) follows since there are more than $2s-r-2$ elements colored by $\mathbb{Z}$ inside $[first(\infty), last(\infty)]$, else $diam(R) \geq 3s-2$. Thus, (i) holds by taking $first_1^s(\infty)$ and $first_{s+1}^{s+r-1}(\infty) \cup last(\infty)$ as the s and r-element ∞-monochromatic sets, respectively. If there does not exist an element of R before $first_s(\infty)$, condition (ii) holds by taking $first_1^s(\infty)$ and the r-element zero-sum mod r set of diameter at least $\lceil \frac{15s-1}{6} \rceil - 4$. Therefore, $first(R) < first(\infty)$.

If $first(\infty) > 2s-1$, then condition (ii) will follow since we can create an s-element zero-sum mod s set from $first_1^{2s-1}(\mathbb{Z})$, and an r-element ∞-monochromatic set with diameter at least $2s-2$. Hence, $first(\infty) \leq 2s-1$. Within the set R, there are at least $\lceil \frac{15s-1}{6} \rceil - 3$ elements colored by $\mathbb{Z}$. After $first(\infty)$ there can be at most $2s-r-2$ elements colored by $\mathbb{Z}$ in $[first(\infty), last(\infty)]$. Suppose $first(\infty) \leq 2s-2$. This implies that there are at least $\lceil \frac{15s-1}{6} \rceil - 3 - (2s-3) > 2s-r-2$ elements colored by $\mathbb{Z}$ in $[first(\infty), last(\infty)]$, which produces a contradiction. Hence $first(\infty) = 2s-2$. If there are three or more residues mod s in $[1, 2s-2]$, then there exists an s-element zero-sum mod s set in $[1, 2s-2]$ and condition (ii) follows. Hence there are two residues mod s in $[1, 2s-2]$, each with $s-1$ elements. By the pigeonhole principle, one such residue must be color c mod r. By fixing the first element not colored by $last(\mathbb{Z})$ within $[1, 2s-2]$ and $last(\mathbb{Z})$ (or $last_2(\mathbb{Z})$ if there is only one residue not colored c or d), we obtain a contradiction using Theorem 2.3 via condition (i) (or other conditions that we will discuss later hold). Therefore, we will assume there are at least r elements colored by c mod r.

Case 3.3: Thus, there exists a color $c \in \mathbb{Z}_r$ such that $|\Delta^{-1}(c) \cup S| \geq \lfloor \frac{5s-1}{3} \rfloor - 1$ (r). If $r > \lfloor \frac{5s-1}{3} \rfloor - 1$, let $\gamma = r - (\lfloor \frac{5s-1}{3} \rfloor - 1)$. If $r - (\lfloor \frac{5s-1}{3} \rfloor - 1)$ is negative, then $\gamma = 0$. Suppose that there are at least $\lceil \frac{7s+1}{3} \rceil - 2$ $(\lceil \frac{7s+1}{3} \rceil - 2 + \gamma)$ elements colored by ∞. If there are no elements colored by c before $first_m(\infty)$ then condition (ii) holds since there exists an s-element ∞-monochromatic set A with $diam(A) \leq \lfloor \frac{5s-1}{3} \rfloor - 2$ $(\lfloor \frac{5s-1}{3} \rfloor - 2 - \gamma)$ and an r-element c-monochromatic set of diameter at least $\lfloor \frac{5s-2}{3} \rfloor - 1$. If $first(\infty) > 2s-1$, we obtain a contradiction since we can create an s-element zero-sum mod s set from $first_1^{2s-1}(\mathbb{Z})$ and an r-element ∞-monochromatic set with a diameter at least $2s-2$, namely $first_1^{r-1}(\infty) \cup last(\infty)$.

Hence, $first(\infty) \leq 2s-1$. Also, notice that there are at most $\lfloor \frac{2s-1}{3} \rfloor - 1$ $(\lfloor \frac{2s-1}{3} \rfloor - 1 - \gamma)$ holes in $[first(\infty), last(\infty)]$, else condition (i) follows in color ∞. If there are elements of c both before and after $[first(\infty), last(\infty)]$, condition (i) follows since there are at least $\lceil \frac{7s-1}{3} \rceil - 2$ $(\lceil \frac{7s-1}{3} \rceil - 2 + \gamma)$ holes colored by ∞. If there are more than s elements colored by c before $first(\infty)$, condition (ii) follows, so there must be fewer than s elements

colored by c before $first(\infty)$. thus, there exists at least one element colored by c after $last(\infty)$. Hence, all the elements colored by c occur after $first(\infty)$ else condition (i) follows via $C = first_1^{r-1}(c) \cup last(c)$. Since there are at least s elements colored by c after $last(\infty)$, there can be no elements colored by c before $first_s(\infty)$, else condition (i) holds since $last(c) - first(c) \geq 3s - 2$. But then condition (ii) holds by taking an s-element ∞-monochromatic set and an r-element c-monochromatic set, which is an r-element zero-sum mod r set. Thus, there are at most $\lceil \frac{7s+1}{3} \rceil - 3$ ($\lceil \frac{7s+1}{3} \rceil - 3 + \gamma$) colored by ∞. This implies that there are at least $\lfloor \frac{6s-1}{3} \rfloor - 1$ ($r + \lfloor \frac{s}{3} \rfloor$) elements colored by c.

Now suppose there are at least $\lceil \frac{6s+1}{3} \rceil - 2$ ($\lceil \frac{6s+1}{3} \rceil - 2 + \gamma$) elements colored by ∞. We can then make the same argument as above to obtain a contradiction. If $first(\infty) > 2s - 1$, we obtain a contradiction since we can create an s-element zero-sum mod s from $first_1^{2s-1}(\mathbb{Z})$, and an r-element ∞-monochromatic set with a diameter of at least $2s - 2$. Hence $first(\infty) \leq 2s-1$. Also notice that there are at most $\lfloor \frac{3s-1}{3} \rfloor - 1$ ($\lfloor \frac{3s-1}{3} \rfloor - 1 - \gamma$) holes in $[first(\infty), last(\infty)]$, else condition (i) follows. If there are elements of c both before and after $[first(\infty), last(\infty)]$, condition (i) follows since there are at least $\lceil \frac{6s+1}{3} \rceil - 2$ ($\lceil \frac{6s+1}{3} \rceil - 2 + \gamma$) holes colored by ∞. If there are more than s elements colored by c before $first(\infty)$, condition (ii) follows. Hence, there must be fewer than s elements colored by c before $last(\infty)$. Hence, all the elements colored by c occur after $first(\infty)$. Since there are at least s elements colored by c after $last(\infty)$, there can be no elements colored by c before $first_s(\infty)$, else condition (i) holds since $last(c) - first(c) \geq 3s - 2$. Hence, condition (ii) holds by taking an s-element ∞-monochromatic set and an r-element c-monochromatic set, which is an r-element zero-sum mod r set. Therefore, there are at most $\lceil \frac{6s+1}{3} \rceil - 3$ ($\lceil \frac{6s+1}{3} \rceil - 3 + \gamma$) elements colored by ∞. This implies there are at least $\lfloor \frac{7s-1}{3} \rfloor - 1$ ($r + \lfloor \frac{2s}{3} \rfloor$) elements colored by c.

Next, suppose there are at least $\lceil \frac{5s+2}{3} \rceil - 2$ ($\lceil \frac{5s+2}{3} \rceil - 2 + \gamma$) elements colored by ∞. If there are at least $\lfloor \frac{s-1}{3} \rfloor + 1$ ($\lfloor \frac{s-1}{3} \rfloor + 1 - \gamma$) elements colored by $\mathbb{Z}$ inside $[first(\infty), last(\infty)]$, then we proceed in exactly the same manner as the previous two paragraphs. Hence, there are at most $\lfloor \frac{s-1}{3} \rfloor$ ($\lfloor \frac{s-1}{3} \rfloor - \gamma$) elements colored by c in $[first(\infty), last(\infty)]$. In this case, it still follows that elements colored by c can not be both before $first(\infty)$ and after $last(\infty)$, else condition (i) holds. Also, if the elements of c are after $last(\infty)$, condition (ii) follows by selecting an s-element ∞-monochromatic set from $[first(\infty), last(\infty)]$ and an r-element c-monochromatic set with the elements colored by c after $last(\infty)$. Hence, there are at least $2s$ ($r + \lfloor \frac{s}{3} \rfloor$) elements colored by c before $first(\infty)$. Let α be the number of elements of $\mathbb{Z}$ not colored by c in the minimal diameter s-element zero-sum mod s set in $[first(c), last(c)]$. Call this minimal-diameter s-element zero-sum mod s set K of diameter at most $t = s - 1 + \alpha + \frac{s}{(r,s)} - 1$. If this diameter is less than or equal to $last(\infty) - first(\infty)$, then we are done. Otherwise, we know that $t \geq \lfloor \frac{5s-1}{3} \rfloor - 2$. Notice that $last(\infty) - first(\infty) \leq 2s - 3$, else condition (ii) holds and there are at most $3s - 3$ elements colored by c mod r. This means there are at least $s + \alpha + \frac{s}{(r,s)} - 1$ elements colored by $\mathbb{Z}$ but not c mod r. But we know that $\alpha + \frac{s}{(r,s)} - 1 \geq \lfloor \frac{2s-1}{3} \rfloor - 1$. Thus, there are at least r elements colored by $\mathbb{Z}$ not colored by c, and we have already considered this case. Thus, there are at least $\lfloor \frac{8s-1}{3} \rfloor - 1$ ($\lfloor \frac{8s-1}{3} \rfloor - 1 + \gamma$) elements colored by c and there are at most $\lceil \frac{5s+1}{3} \rceil - 3$ ($\lceil \frac{5s+1}{3} \rceil - 3 + \gamma < r$) elements colored by ∞. This contradicts our assumption that $\Delta^{-1}(\infty)| \geq r$. Thus, we are finished with this case.

Therefore, we can now apply Theorem 2.3 to observe that either the lemma is satisfied or all but at most $a - 2$ of the elements colored by $\mathbb{Z}$ are colored by elements from the

same coset $(a\mathbb{Z}_m + \alpha)$ of $\mathbb{Z}_m$. In the latter case, Theorem 2.2 implies that any subset of cardinality $(m + \frac{m}{a} + a - 3) \leq \lceil \frac{3s}{2} \rceil - 1$ must contain an m-element zero-sum mod m set.

If $r \geq 2s - 2$, condition (i) follows with the r-element zero-sum mod r set G' constructed from the set $G = first_1^{\lceil \frac{r}{2} \rceil}(\mathbb{Z}) \cup last_1^{r-1}(\mathbb{Z})$. If there are at least $\lceil \frac{r}{2} \rceil$ elements colored by $\mathbb{Z}$ before $first(\infty)$, then condition (i) or (ii) follows. Condition (i) follows if $last_{\lceil \frac{r}{2} \rceil}(\mathbb{Z}) \geq 3s - 3$, since G' has a diameter of at least $3s - 3$. However, if this is not the case, then there are at most $\lceil \frac{r}{2} \rceil - 1$ integers colored by $\mathbb{Z}$ in the interval $[3s - 2, \max(last(\mathbb{Z}), last(\infty))]$. Hence, $last(\infty) \geq 6s - 5 + \frac{s}{(r,s)} - 1 - \lceil \frac{r}{2} \rceil$ $(last(\infty) \geq 2r - 1 + \omega - \lceil \frac{r}{2} \rceil)$, and therefore $first(\infty) \geq 3s - 2 - \lceil \frac{r}{2} \rceil$ $(3r - 2s - 2 - \delta - \lceil \frac{r}{2} \rceil)$. Hence, the first $\lceil \frac{3s}{2} \rceil$ elements of $[1, \max(last(\mathbb{Z}), last(\infty))]$ are colored entirely by $\mathbb{Z}$, and thus there exists an s-element zero-sum mod s set H with $diam(H) \leq \lceil \frac{3s}{2} \rceil - 2$. Hence, condition (ii) holds. Therefore, $first(\infty) \leq \lceil \frac{r}{2} \rceil$. Thus, the interval $[3s + \lceil \frac{r}{2} \rceil - 2, \max(last(\mathbb{Z}), last(\infty))]$ is colored entirely by $\mathbb{Z}$. Suppose there are fewer than s elements colored by $\mathbb{Z}$ between $[first(\infty), last(\infty)]$. Let $A_1 = [1, first(\infty) - 1], A_2 = last_1^{\lceil \frac{r}{2} \rceil - first(\infty)+1}(\mathbb{Z}, [last(\infty), 4s - 3])$ $(last_1^{\lceil \frac{r}{2} \rceil - first(\infty)+1}(\mathbb{Z}, [last(\infty)\ , \lceil \frac{2s}{3} \rceil + 2r - 3 - \delta]))$, and $A_3 = [6s - r - 5, \max(last(\mathbb{Z}), last(\infty))]$ $([r - 1 + \omega, \max(last(\mathbb{Z}), last(\infty))])$. In this case, we are guaranteed that $A = A_1 \cup A_2 \cup A_3$ contains an r-element zero-sum mod r set. Condition (i) holds if the set consists of elements from A_1 and A_3, or A_1, A_2 and A_3. An r-element zero-sum mod r set can not be constructed using only A_1 and A_2. If an r-element zero-sum mod r set appears in A_2 and A_3 only, then condition (ii) follows since there exists an s-element ∞-monochromatic set B with $diam(B) \leq 2s - 2$ using the first s elements colored by ∞, and there exists an r-element zero-sum mod r set B' with $diam(B') \geq 2s - 1$. Thus there must be at least s elements colored by $\mathbb{Z}$ in $[first(\infty), last(\infty)]$. This implies there are at most $2s - 3$ elements colored by ∞ else condition (i) holds. Thus, there are at least $4s - 2(2r - 2s + 2 + \omega)$ elements colored by $\mathbb{Z}$. In this case, the r-element zero-sum mod r set A' as described before, has a diameter of at least $3s - 3$, and thus condition (i) follows. We have a contradiction, and the proof is complete. $\square$

Theorem 4.8. *Let r and s be positive integers where $r \geq s \geq 3$. If $r \leq \lfloor \frac{5s-1}{3} \rfloor - 1$, then $f(s, r, \{\infty\} \cup \mathbb{Z}) = 9s - 7 + \frac{s}{(r,s)} - 1$.*

Proof. The coloring $\Delta : [1, 9s - 8 + \frac{s}{(r,s)} - 1] \longrightarrow \{\infty\} \cup \mathbb{Z}$ given by the string:

$$10^{s-1}1^{s-2}\infty^{s-1}10^{s-1}2^{\frac{s}{(r,s)}-1}0^{s-1}1^{2s-2}\infty^{2s-2} \bmod s$$
$$10^{s-1}1^{s-2}\infty^{s-1}10^{2s-2+\frac{s}{(r,s)}-1}1^{2s-2}\infty^{2s-2} \bmod r$$

implies that $f(s, r, \{\infty\} \cup \mathbb{Z}) > 9s - 8 + \frac{s}{(r,s)} - 1$. Next we show that $f(s, r, \{\infty\} \cup \mathbb{Z}) \leq 9s - 7 + \frac{s}{(r,s)} - 1$. Let $\Delta : [1, 9s - 7 + \frac{s}{(r,s)} - 1] \longrightarrow \{\infty\} \cup \mathbb{Z})$ be an arbitrary coloring. By Theorem 1.1, the interval $[1, 3s - 2]$ contains an s-element zero-sum mod s subset S such that $diam(S) \leq 3s - 3$. Shifting the interval $[3s - 1, 9s - 7 + \frac{s}{(r,s)} - 1]$ to the interval $[1, 6s - 5 + \frac{s}{(r,s)} - 1]$ and applying Lemma 4.7 completes the proof for $r \geq s \geq 3$. $\square$

Theorem 4.9. *Let r and s be positive integers where $r \geq s \geq 3$. If $\lfloor \frac{5s-1}{3} \rfloor - 1 \leq r \leq 3s-3$, then $f(s, r, \{\infty\} \cup \mathbb{Z}) = 3s + 2r - 3 + \min(3s - 3, s + r - 1 + \frac{s}{(r,s)} - 1)$.*

Proof. The coloring $\Delta : [1, 3s+2r-4+\min(3s-3, s+r-1+\frac{s}{(r,s)}-1)] \longrightarrow \{\infty\} \cup \mathbb{Z}$ given by the string:

$$10^{s-1}1^{s-2}\infty^{s-1}10^{s-1}2^{\frac{s}{(r,s)}-1}0^{\min(r,2s-1-\frac{s}{(r,s)})}1^{r-1}\infty^{r-1} \bmod s$$
$$10^{s-1}1^{s-2}\infty^{s-1}10^{\min(3s-3,s+r-1+\frac{s}{(r,s)}-1)}1^{r-1}\infty^{r-1} \bmod r$$

implies that $f(s, r, \{\infty\} \cup \mathbb{Z}) > 3s+2r-4+\min(3s-3, s+r-1+\frac{s}{(r,s)}-1)$. Next we show that $f(s, r, \{\infty\} \cup \mathbb{Z}) \leq 3s+2r-3+\min(3s-3, s+r-1+\frac{s}{(r,s)}-1)$. Let $\Delta : [1, 3s+2r-3+\min(3s-3, s+r-1+\frac{s}{(r,s)}-1)] \longrightarrow \{\infty\} \cup \mathbb{Z})$ be an arbitrary coloring. By Theorem 1.1, the interval $[1, 3s-2]$ contains an s-element zero-sum mod s subset S such that $diam(S) \leq 3s-3$. Shifting the interval $[3s-1, 3s+2r-3+\min(3s-3, s+r-1+\frac{s}{(r,s)}-1)]$ to the interval $[1, 2r-1+\min(3s-3, s+r-1+\frac{s}{(r,s)}-1)]$ and applying Lemma 4.7 completes the proof for $r \geq s \geq 3$. □

Notice that for these two cases, we have a generalization in the sense of Erdős-Ginzburg-Ziv if and only if $(r, s) = 1$ or $3s-3 \geq s+r-1+\frac{s}{(r,s)}-1$. As we can see in the next theorem, for $r > 3s-3$ we always obtain a generalization in the sense of Erdős-Ginzburg-Ziv.

Theorem 4.10. *Let r and s be positive integers where $r \geq s \geq 2$. If $r > 3s-3$, then $f(s, r, \{\infty\} \cup \mathbb{Z}) = 3s+3r-4$.*

Proof. The coloring $\Delta : [1, 3s+3r-5] \longrightarrow \{\infty\} \cup \mathbb{Z}$ given by the following string, which is the same mod s or mod r:

$$\infty 1^{s-1}\infty^{s-2}0^{s-1}\infty 1^{r-1}\infty^{r-1}0^{r-1}$$

implies that $f(s, r, \{\infty\} \cup \mathbb{Z}) \geq 3s+3r-4$. By the pigeonhole principle, it follows that the interval $[1, 3s-2]$ contains an s-element zero-sum mod s or ∞-monochromatic subset S_1 with $diam(S_1) \leq 3s-3$, and that the interval $[3s-1, 3s+3r-4]$ contains an r-element zero-sum mod r or ∞-monochromatic subset S_2 with $diam(S_2) \geq 3s-3$. Hence S_1 and S_2 complete the proof. □

Acknowledgements. The author would like to thank Arie Bialostocki for supervising this research and to David Grynkiewicz for his helpful communications.

References

1. A. Bialostocki, G. Bialostocki, and D. Schaal, A zero-sum theorem. J. Combin. Theory, Ser A **101** (2003) 147-152.

2. A. Bialostocki, and P. Dierker, On the Erdős-Ginzburg-Ziv theorem and the Ramsey numbers for stars and matchings. Discrete Math. **110** (1992), no. 1-3, 1-8.

3. A. Bialostocki, P. Dierker, D. Grynkiewicz, and M. Lotspeich, On some developments of the Erdős-Ginzburg-Ziv Theorem II. To appear in Acta Arith.

4. A. Bialostocki, P. Erdős and H. Lefmann, Monochromatic and zero-sum sets of nondecreasing diameter. Discrete Math. **137** (1995), 19-34.

5. A. Bialostocki, R. Sabar, On constrained 2-Partitions of monochromatic sets and generalizations in the sense of Erdős-Ginzburg-Ziv. To appear, Ars Combinatorica (2002).

6. A. Bialostocki, R. Sabar, and D. Schaal, On a zero-sum generalization of a variation of Schur's equation. Preprint, (2002).

7. Y. Caro, Zero-Sum problems–a survey. Discrete Math. **152** (1996) 93-113.

8. P. Erdős, A. Ginzburg and A. Ziv, Theorem in additive number theory. Bull. Research Council Israel **10 F** (1961) 41-43.

9. D. Grynkiewicz, On a partition analog of the Cauchy-Davenport Theorem. Preprint, (2002).

10. D. Grynkiewicz, On four color monochromatic sets with nondecreasing diameter. Manuscript, (2002).

11. D. Grynkiewicz, On four colored sets with nondecreasing diameter and the Erdős-Ginzburg-Ziv Theorem. J. Combin. Theory, Ser. A, **100** (2002), no. 1, 44-60.

12. D. Grynkiewicz and R. Sabar, Monochromatic and zero-sum sets of nondecreasing modified diameter and the Erdos-Ginzberg-Ziv theorem. Preprint, (2002).

13. D. Grynkiewicz and A. Schultz, A 5-color Erdős-Ginzberg-Ziv generalization. Manuscript, (2002).

14. D. Grynkiewicz and C. Yerger, On three sets with nondecreasing diameter. Manuscript, (2003).

15. R. Sabar, Monochromatic and zero-sum solutions of a variation of Schur's equation. Manuscript, (2002).

16. R. Sabar, Partial-Ascending waves and the Erdős-Ginzburg-Ziv Theorem. Manuscript, (2002).

17. D. Schaal, On some zero-sum Rado-type problems. Ph.D. dissertation, U. of Idaho (1994).

18. A. Schultz, On a modification of a problem of Bialostocki-Erdős-Lefmann. Manuscript, (2002).

19. B. Wong, A generalization of a problem of Bialostocki, Erdős and Lefmann. Manuscript, (2002).

Received:
Final version received: